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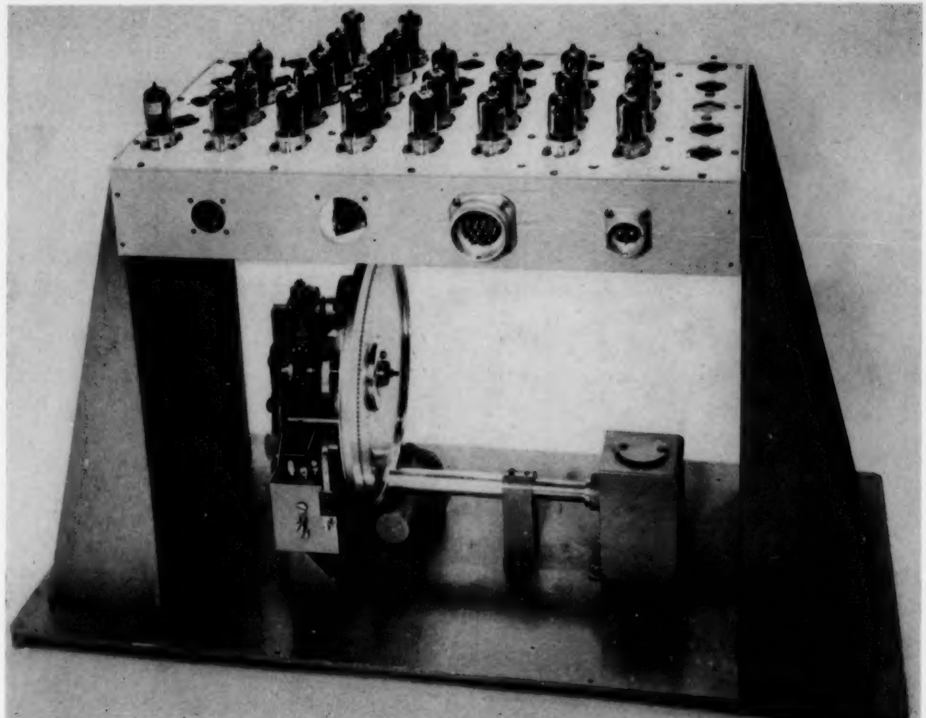

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On a Special Case of a Quadrature Formula of Christoffel

1. This note deals with the numerical evaluation of the integral

$$\int_{-1}^{1} w(x)f(x)dx,$$

where the special case $a = 1$ of the weight function $w(x) = (a^2 + x^2)^{-1}$ is treated.

Consider quite generally the integral

$$\int_{\alpha}^{\beta} w(x)f(x)dx, \tag{1}$$

where for the weight function $w(x)$ we assume only that the moments $\mu_r = \int_{\alpha}^{\beta} x^r w(x)dx$ exist. It is then known[1] that n properly chosen values of the function, $f(x_\nu)$, $\nu = 1, 2, \cdots, n$, suffice to give the integral exactly by the sum $\sum_{\nu=1}^{n} p_\nu f(x_\nu)$, provided $f(x)$ is a polynomial of degree not higher than $2n - 1$. The integration formula thus yields the same accuracy with n ordinates as, e.g., NEWTON–COTES formula using $2n$ equidistant values. The division points $x_\nu (\alpha < x_\nu < \beta)$ are the zeros of certain polynomials $F_n(x)$ of degree n, orthogonal with respect to the weight function $w(x)$. The above quadrature formula was first derived by GAUSS for the special case $w(x) = 1$; $F_n(x)$ is then the LEGENDRE polynomial of degree n. Similar quadrature formulae of the Gaussian type, as we will call them, have been established for the weight functions

$$w(x) = (x - \alpha)^\gamma(\beta - x)^\lambda, \gamma > -1, \lambda > -1,$$
$$w(x) = x^\lambda e^{-x}, \alpha = 0, \beta = \infty, \lambda > -1,$$
$$w(x) = \exp(-x^2), \beta = -\alpha = \infty,$$

where the arguments are the zeros of the JACOBI, LAGUERRE and HERMITE polynomials.[1] As far as the author is aware no quadrature formula associated with the weight function $w(x) = (a^2 + x^2)^{-1}$ has been derived before. It has, therefore, been considered worth while to treat the case $\beta = -\alpha = 1$, for which division points x_ν and weights p_ν are supplied for $\nu = 2(1)7$ and $a = 1$.

The importance of quadrature formulae of the Gaussian type for purely numerical work is admittedly restricted through the irrationality of the arguments which necessitates cumbersome interpolation. For problems connected with the numerical solution of integral or integro-differential equations, however, the Gaussian formulae have proved to be of great value. Considering, e.g., a FREDHOLM integral equation of the second kind

$$f(x) = g(x) + \int_{\alpha}^{\beta} w(y)f(y)K(x, y)dy \tag{2}$$

we have approximately

$$f(x) = g(x) + \sum_{\nu=1}^{n} p_\nu f(y_\nu)K(x, y_\nu). \tag{3}$$

Putting $x = y_\nu$, we can generally solve the system for the unknowns $f(y_\nu)$, and as soon as these basic function values are known, $f(x)$ can be computed from (3) for any value of the argument x. This procedure was first employed for the numerical solution of integral equations by NYSTRÖM.[2] The same procedure of replacing the integral by a weighted sum for numerical solution of integro-differential equations has been utilized by WICK[3] and CHANDRASEKHAR.[4]

2. The natural starting point for establishing the quadrature formulae is the orthogonal property characteristic of the polynomials $F_n(x)$. The following procedure for deriving $F_n(x)$ is due to CHRISTOFFEL.[5] The polynomial $F_n(x)$ is orthogonal with respect to $w(x)$, if

$$\text{(4)}\qquad \int_\alpha^\beta w(x)F_m(x)F_n(x)dx = 0,\ m \neq n;\ m, n = 0, 1, \ldots$$

or

$$\text{(5)}\qquad \int_\alpha^\beta x^k w(x)F_n(x)dx = 0,\ n = 1, 2, \ldots;\ k < n.$$

Defining the quantities

$$\text{(6)}\qquad I_n = \int_\alpha^\beta w(x)F_n^2(x)dx;\ k_nI_n = \int_\alpha^\beta xw(x)F_n^2(x)dx,$$

the following recurrence formula holds for any three consecutive orthogonal polynomials

$$\text{(7)}\qquad F_{n+1}(x) = (x - k_n)F_n(x) - I_nF_{n-1}(x)/I_{n-1}.$$

It is here required that the coefficient of x^n in the polynomial $F_n(x)$ is equal to $+1$.

The construction of the polynomials can thus be performed from (7) as soon as we know F_0 and $F_1(x)$.

The weights are given by

$$\text{(8)}\qquad p_r = \frac{1}{F_n'(x_r)}\int_\alpha^\beta w(x)\frac{F_n(x)}{x - x_r}dx,\ r = 1, 2, \ldots, n.$$

Re-writing this as follows

$$\text{(9)}\qquad p_r = \frac{-1}{F_n'(x_r)F_{n+1}(x_r)}\int_\alpha^\beta w(x)\frac{F_{n+1}(x)F_n(x_r) - F_{n+1}(x_r)F_n(x)}{x - x_r}dx,$$

and making use of the CHRISTOFFEL–DARBOUX identity

$$\text{(10)}\qquad \sum_{\nu=0}^{n} F_\nu(x)F_\nu(y)/I_\nu = [F_{n+1}(x)F_n(y) - F_n(x)F_{n+1}(y)]/[(x - y)I_n]$$

we get

$$\text{(11)}\qquad p_r = \frac{-I_n}{F_n'(x_r)F_{n+1}(x_r)}\sum_{\nu=0}^{n}\frac{F_\nu(x_r)}{I_\nu}\int_\alpha^\beta w(x)F_\nu(x)dx.$$

But from (5) it is clear that all integrals with the exception of $\int_\alpha^\beta w(x)F_n^2dx$ vanish, and consequently the following representation holds.[1]

$$p_r = -\, I_n/[F_n'(x_r)F_{n+1}(x_r)]. \tag{12}$$

Observing that

$$F_{n+1}(x_r) = -\, I_nF_{n-1}(x)/I_{n-1} \tag{13}$$

we obtain the alternative formula

$$p_r = I_{n-1}/[F_n'(x_r)F_{n-1}(x_r)]. \tag{14}$$

3. In the case to be discussed we have $w(x) = (a^2 + x^2)^{-1}$; $\beta = -\alpha = 1$. As the integrand $xw(x)F_n^2(x)$ is an odd function, k_n is zero, and

$$F_n(x) = xF_{n-1}(x) - I_{n-1}F_{n-2}(x)/I_{n-2}, \tag{15}$$

which can be used for constructing $F_n(x)$. Christoffel[5] has communicated the following general expression for $F_n(x)$

$$F_n(x) = P_n(x) - \sigma_nP_{n-2}(x)/\sigma_{n-2}, \tag{16}$$

where $P_n(x)$ is the Legendre polynomial of degree n, and

$$\sigma_n = \sigma_n(ai) = 2W_{n-1}(ai) + 2iP_n(ai)\operatorname{arccot} a, \tag{17}$$

i being the imaginary unit. The polynomial $W_{n-1}(u)$ of degree $n - 1$ is expressed by the formula

$$W_{n-1}(u) = \frac{2n-1}{1\cdot n}P_{n-1}(u) + \frac{2n-5}{3(n-1)}P_{n-3}(u) + \frac{2n-9}{5(n-2)}P_{n-5}(u) + \dots . \tag{18}$$

For $P_n(x)$ and associated functions reference is made to E. W. Hobson.[6] The quantities $\sigma_n(i)$ and $P_n(i)$ are listed in Table 1 for $n = 0(1)7$.

Table 1

n	$\sigma_n(i)$	$P_n(i)$	I_n
0	$\frac{1}{2}\pi i$	1	—
1	$\frac{1}{2}(4 - \pi)$	i	$\frac{1}{2}(4 - \pi)$
2	$(3 - \pi)i$	-2	$8(\pi - 3)/(3\pi)$
3	$(6\pi - 19)/3$	$-4i$	$8(19 - 6\pi)/45(4 - \pi)$
4	$(51\pi - 160)i/12$	$17/2$	$8(51\pi - 160)/1575(\pi - 3)$
5	$(1744 - 555\pi)/60$	$37i/2$	$8(1744 - 555\pi)/11025(19 - 6\pi)$
6	$(644 - 205\pi)i/10$	-41	$512(205\pi - 644)/121275(51\pi - 160)$
7	$(1610\pi - 5058)/35$	$-92i$	$1024(5058 - 1610\pi)/693693(1744 - 555\pi)$

The division points $-1 < x_r < +1$ are thus the roots of the equation

$$F_n(x) = P_n(x) - (\sigma_n/\sigma_{n-2})P_{n-2}(x) = 0. \tag{19}$$

We observe the following properties of $F_n(x)$: The quotient σ_n/σ_{n-2} is always real, the zeros of $F_n(x) = 0$ are distributed as the zeros of $P_n(x) = 0$, and

consequently symmetrical with respect to $x = 0$. For n odd, $x = 0$ satisfies (19).

Inserting (16) in (6) we obtain an explicit expression of I_n. As the coefficient of x^n differs from $+1$ we put

$$I_n = c_n \int_{-1}^{1} w(x) P_n^2(x)dx, \tag{20}$$

where $w(x) = (a^2 + x^2)^{-1}$. We get

$$I_n c_n^{-1} = \int_{-1}^{1} (a^2+x^2)^{-1} P_n^2(x)dx + (\sigma_n/\sigma_{n-2})^2 \int_{-1}^{1} (a^2+x^2)^{-1} P_{n-2}^2(x)dx \tag{21}$$
$$- 2(\sigma_n/\sigma_{n-2}) \int_{-1}^{1} (a^2+x^2)^{-1} P_n(x) P_{n-2}(x)dx.$$

The first and last of these integrals can be evaluated as follows:

$$\int_{-1}^{1} (a^2 + x^2)^{-1} P_n^2(x)dx = -i\sigma_n P_n(ai)/a \tag{22}$$

and

$$\int_{-1}^{1} (a^2 + x^2)^{-1} P_n(x) P_{n-2}(x)\, dx = -i\sigma_n P_{n-2}(ai)/a. \tag{23}$$

Introducing (22) and (23) in (21) we find

$$I_n c_n^{-1} = -i(\sigma_n/\sigma_{n-2})[\sigma_{n-2} P_n(ai) - \sigma_n P_{n-2}(ai)]/a, \tag{24}$$

which also can be written as follows, if we insert (17) in the bracket,

$$I_n c_n^{-1} = -2i(\sigma_n/\sigma_{n-2})[W_{n-2}(ai) P_n(ai) - W_{n-1}(ai) P_{n-2}(ai)]/a. \tag{25}$$

For c_n we have the following expression

$$c_n = [n!/(1\cdot 3 \ldots (2n-1))]^2. \tag{26}$$

TABLE 2

n	$\pm x_\nu$	p_ν
2	0.5227232	0.5000000
3	0	0.5061318
	0.7438173	0.2469341
4	0.3151531	0.3583405
	0.8445005	0.1416595
5	0	0.3368260
	0.5132821	0.2404139
	0.8965229	0.09117313
6	0.2253363	0.2700545
	0.6410942	0.1664560
	0.9264497	0.06348956
7	0	0.2521628
	0.3888781	0.2065587
	0.7263167	0.1206198
	0.9451281	0.04674011

The quantities I_n for $a = 1$ are given in Table 1 while arguments and weights computed for $n = 2(1)7$ are listed in Table 2.

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[1] G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc., *Colloquium Publs.*, v. 23, 1939, chap. III & XV.

[2] E. J. Nyström, "Über die praktische Auflösung von Integralgleichungen mit Anwendungen auf Randwertaufgaben," *Acta Math.*, v. 54, 1930, p. 185–204. Integral equations defined within an infinite interval have been discussed from the numerical point of view by A. Reiz, "On the numerical solution of certain types of integral equations," *Arkiv Mat. Astr. Fysik*, v. 29A, no. 29, 1943, 21 p.

[3] G. C. Wick, "Über ebene Diffusionsprobleme," *Z. Physik*, v. 121, 1943, p. 702–718.

[4] S. Chandrasekhar, "On the radiative equilibrium of a stellar atmosphere, II," *Astrophys. Jn.*, v. 100, 1944, p. 76–86 and following volumes; see also the same author's *Radiative Transfer*, Oxford Univ. Press, 1950.

[5] E. B. Christoffel, "Sur une classe particulière de fonctions entières et de fractions continues," *Annali di Mat.* S.2, v. 8, 1877, p. 1–10.

[6] E. W. Hobson, *The Theory of Spherical and Ellipsoidal Harmonics*, Cambridge Univ. Press, 1931, particularly chap. II: § 34.

New Information Concerning Isaac Wolfram's Life and Calculations

1. Introductory.—We have already noted certain items[1–4] regarding Wolfram, an eighteenth century Dutch artillery officer, one of his mathematical tables, and his contacts with Lambert.[5] Hardly anything else is to be found in any mathematical history, periodical, or bibliography—to practically all of which we refer. No one has previously remarked that in two volumes of correspondence of Lambert, edited by one of the Bernoullis (1784, 1785–87), more than 200 pages of material,[9] including many of Wolfram's letters, tell us much concerning him and his mathematical activities for over 35 years. In what follows my main objects shall be to give some idea of the nature of the new material, and also to include some interesting recently discovered additional facts, supplied by J. H. B. Kemperman, research worker at the Mathematisch Centrum, Amsterdam. Bernoulli refers to the "great calculator Wolfram"[9] (v. 5, p. 464), who was notified that the Prussian Academy of Sciences would be glad to receive as a legacy the complete collection of his logarithmic calculations for preservation in its archives.

2. Mr. Kemperman's Report[29] (24 January 1950).—"According to information received from the Royal Military Academy at Breda and the General Public Record Office at The Hague, the following data are certain—

(i) His full name is Isaac Wolfram (according to an army list from 1781). He is always indicated by the name J. Wolfram.

(ii) Before 1747 Wolfram was not yet an officer.

(iii) On August 3, 1747 he became "onderlieutenant" (artillery).

(iv) On November 27, 1764 he was promoted to be "ordinairlieutenant."

(v) On September 1, 1779 he became "captain-lieutenant."

(vi) Before 25 August 1788 Wolfram was no longer in the army, for on that date his substitute A. van Hoey van Oostee was sent out (according to the consignment book of the Council of State). In the army-list of 1786 Wolfram's name still appears, but in that of 1789 it is no longer found.

(vii) In 1778 he was stationed at Nijmegen.

(viii) About 1780 he was stationed among other places at Axel, Steenbergen, Hellevoetsluis, and Brielle.

From the artillery regiment in which J. Wolfram served, there are no confidential reports left, nor other documents from which the date of his birth or his birthplace might appear. The archivist of the town of Nijmegen could not find any data concerning Wolfram."

3. Comments.—In July and September 1774 Wolfram was in Danzig and in each of these months wrote letters from there to Lambert[9] (XXVII, XXX). In the first letter he refers to Danzig as his "Vaterstadt," p. 509. Writing to Lambert from Nimwegen (so the name is always spelled in the German correspondence; the English form is Nimeguen) in 1776 (XXXII) he tells us that his factor table, up to 126000 (giving the least divisors greater than 5, of the numbers), was made at Danzig in 1743 and enlarged to 300000 in Holland.

The postscript to a letter of 6 September 1774 to Lambert[9] (XXX) (in translation of the German original) is as follows: "Should the Royal Prussian ambassador at The Hague, Herr von Thulemayer, mention my name to the Prince of Orange, it might well happen that I should not have to wait 17 years and some months, as I did in 1764, when I became "Ordinairlieutenant"; that I was in the campaigns of 1746 and 1747, and did my duty in the battles of Roucoux and Laffeld [places in Belgium], has doubtless been long since forgotten." As we have seen, it was not until five years after this that Wolfram received his promotion.

Wolfram was an officer in 1747. On the assumption that he was then 22 years old (recall that 4 years earlier he had completed a factor table) he was born about 1725 and did not leave Danzig for Holland before 1743. The accuracy of Mr. Kemperman's statement "no longer in the army," in August 1788, is readily checked, since we know that Wolfram was dead by 1787. Writing in this year[9] (v. 5, p. 344) Bernoulli tells us that "only in recent months have I learned with certainty that this industrious and skillful calculator, this really extraordinary man in his field, is no longer living. But I am without particulars of time, place, etc., where his life ended, or what may have been the fate of his many painstakingly handwritten calculations, as one must very much desire to learn, when one recalls from his letters to Lambert all his labors and intended plans."

4. First Publication, *et al.*—Wolfram's first mathematical publication seems to have been a portion of his 1743 factor table[17,20] (3), namely: "Proeve van eene tafel ter ontledinge der getallen" [specimen of a table for the factorization of numbers], which appeared in Hollandse Maatschappy de Wetenschappen, Haarlem, *Verhandelingen*, v. 2, 1755, p. 622–624. This table gives the least divisors greater than 5 for numbers less than 6000. After three pages of text the three-section table is on a folding plate 16.7 × 44 cm. in size. Although the rest of the table up to 126000 was promised by the editor for publication in the next volume (1757), the promise was never kept. In a letter to Lambert[9] in 1772 (XX), p. 453, Wolfram refers to the matter and quotes the reason for nonpublication of the rest of his table, as given in a letter dated 6 March 1761, from VAN DER AA, the secretary of the Society; in translation from the Dutch this passage is as follows: "because one can easily find the divisors of a number, when they exist, by means of artifices of VAN SCHOOTEN, KRAFFT, EULER, etc." Wolfram then wrote that

this judgment did not in the least deter him from extending his table, "Vollständige Zergliederung der ersten 300000 Zahlen in einem kurzen Begriffe: oder eine Tafel der kleinste Factor, wenn er grösser als 5, zu finden ist." By 1756 he had already laid plans for extending his table yet further, to a million.

That the scientific editing of the above-mentioned Haarlem publication of 1755 was not of a very high order was also illustrated by two tables of W. O. REITZ in a long-drawn-out article "De berekening van Kunsttallen," p. 166–224 + 4 pages. The tables are of log N, $N = [101(1)1000;18D]$, Δ, and of log $(1 + a \cdot 10^{-n})$, $a = 1(1)9$, $n = [3(1)18;18D]$, with Δ up to $n = 9$. In a letter to Lambert[9] in 1773 (XXII), p. 483, Wolfram states that van der Aa told him in 1759 that the Society had awarded a prize for this paper. Wolfram then points out that the first table has only four more decimal places than that of BRIGGS, and that the logs of 16 of the prime numbers and 9 of the composite numbers, are erroneous. I checked the accuracy of this last statement. Long before this table was published ABRAHAM SHARP gave (1717) log N, for $N = [1(1)100$ and primes to $1100; 61D]$, every logarithm apparently correct to at least 57D.

5. Lambert and his first contact with Wolfram.—During the years 1756–1758 Lambert spent more than a year at the Dutch Universities of Utrecht and Leyden. But, as we have already noted,[3] from 1764 until his death 25 September, 1777, he was almost wholly at Berlin, where he received many favors from Frederick the Great, and was in 1765 elected a fellow of the Royal Academy of Sciences and Belles-Lettres. In this year he published the first part of his *Beyträge*,[6] but the two sections of the second part (which has more immediate interest for our inquiry) came out in 1770. Shortly afterwards, in the same year, his valuable volume of tables, *Zusätze*,[7] appeared. Among the *Beyträge* items is a complete factor and prime table up to 10200 (p. 42–53), and the evaluation of π^{-1} by continued fractions (p. 156–158). In the *Zusätze* there is a table similar in results to Wolfram's, up to 102000, with a separate table of primes; an appeal was made for more extended computations.

On 5 March 1772 Wolfram wrote his first letter to Lambert[9] (XVIII). He reported that on the previous 28 February he had acquired a copy of the *Zusätze* and noted there the request for more extensive factor tables. Hence he described: (a) his own factor table up to 300000 on 25 folio pages; and then also mentioned (b) his 39D common logarithmic table, based on 42D computation, of all primes less than 10000; and (c) his recomputation of J. P. BUCHNER, *Tabula Radicum, Quadratorum & Cuborum ad Radic. 12000 extensa*, Nuremberg, 1701, since its tables of squares and cubes contained so many errors. W. offered to make this material available to L. for publication if he so desired and if L's. exact address were furnished. W. added that he would then send also a list of 70 errors in L's. factor table which he noted during the few days he had the *Zusätze*.

W. then goes on to report that, in the logarithms of A. SHARP in SHERWIN's *Mathematical Tables*, 1726 [second edition], except for the first 16 digits of log 131, and the first 20 digits of log 163, all other digits of these logarithms are incorrect. He also states that in other logarithms of this table one or four or seven digits are incorrect. This is the table on p. 28–29 of the logarithms of 1(1)100 and of all prime numbers less than 200, 51–61D. In

his *Geometry Improv'd*, London, 1717, p. 56–60, Sharp gave a table of the logarithms of [1(1)100, primes less than 1100; 61D]. In this table, for Sherwin's 1726 range, only two errors have been found, namely, unit errors in the 61st decimal places for log 127, and log 149. If Sherwin was authorized to use Sharp's name and computations, it is not a little curious that the 1726 Sherwin, published nine years after the Sharp volume, is so inaccurate and incomplete.[32] The third carefully revised and corrected edition was in 1741, (also 1742) and the fourth in 1761.

W's. great interest in extended logarithmic computation led him to compute logarithms of primes from 63D to 88D,[9] v. 5, p. 460. There are several later references to the Sharp table in Sherwin's tables[9]: v. 4, p. 451, 482, 485–486, 493–496, 518, 522; v. 5, p. 462–463. W. had gained access to the third (1742) edition, containing Sharp's complete table as given in 1717, and completed the checking of this table in October 1780. In a letter dated 2 March 1781, W. wrote that he found Sharp's table "correct except for a few digits in the 61st. decimal place." As a matter of fact, however, there were, apparently, more serious errors in the logarithms of 227, 839, 1009, and also of 643 (1761 Sherwin) where a printer's slip has 42, instead of 24, in the 18th and 19th decimal places. I do not have the third edition for checking.[32]

W. reported also that he had found 8 erroneous values in the 100 logarithms of the "*Clausbergischen Rechenkunst.*" This is C. VON CLAUSBERG, *Demonstrative Rechenkunst*, Leipzig, 1732, which I have not seen; there were also later editions by C. A. HAUSEN, second, 1748; third, 1767; fourth 1771 (see XXIV, p. 495); also p. 453, 489.

And in concluding his first letter to L., W. refers to his discovery in 1754 of a numerical series for π by means of which he computes π correct to 9D. The series is simply Newton's expansion of arcsin x, when $x = \frac{1}{2}$.

6. Lambert's Reply to Wolfram's First Letter and Later Correspondence.—Lambert replied to W. promptly and cordially. He referred to having received, in response to his appeal, several other contributions of tabular material, including one table by "Ober-Finanz Buchhalter Oberreit" of Dresden, which contained the complete factorization of the first half-million numbers; the author planned to complete the table for the first million. In September 1770 L. published a note about this table in order to prevent useless work by others. L. states that in his table collection, planned for some future date, he may possibly use W's. tables of logarithms and of squares and cubes. He also mentions that for a time one of his Berlin friends had been calculating a table of hyperbolic logarithms to many places of decimals, but had put the work aside. W. had already in 1772 commenced the preparation of his great table of hyperbolic logarithms. L. naturally stated that he would be glad to receive lists of errata in his work; he noted various corrigenda reported to him by others. L. suggested that W. should make a table of $u = \ln \tan (45° + \frac{1}{2}w)$, for $w = 45°(1') 90°$, instead of at interval 1° as in Table 32 of the *Zusätze*.

W's. reply[9] (XX) occupies 24 printed pages. He thanked L. for his suggestion as to a computation; sent L. not only the 70 errors in *Zusätze*, referred to earlier (printed in *Beyträge*,[9] v. 3), but also at least 28 more given in this letter; sent also 29 errors in *Beyträge* factor table (5). Among these corrections were a number in the *Zusätze*, Table 2 of primes, in the value for $(\frac{1}{6}\pi)^{\frac{1}{2}}$, and for convergents of the continued fraction for π^{-1}. Here W. carried

the work further even to the extent of WALLIS [*MTAC*, v. 2, p. 340–341; v. 3, p. 172]. W. explained also, with numerical examples, his method (discovered in 1756) for finding the cube root of large numbers, and in particular for small ones, 1(1)125, values, which were necessary on artillery bore sticks.[34] W. found also in terms of square roots the length of sides of regular n-sided polygons, $n = 4(4)24(8)48,15,20(10)40(20)80,120$, and therefrom finds sides for n up to $n = 512$, in order to get approximations to π. In this connection W. refers to T. SIMPSON, *Élémens de Géométrie*, Paris, 1755. Elsewhere in XX he quotes SARGANECK, *Geometrie in Tabellen*,[30], C. MACLAURIN, *Traité des Fluxions*, Paris, 1749, KÄSTNER'S, *Anfangsgründe*,[31] and works of C. v. Clausenberg, L. Euler, G. W. Krafft, A. F. MARCI, J. PELL, W. O. Reitz, F. v. Schooten, A. Sharp. Throughout the correspondence there are many similar references.

And so the correspondence between W. and L. continued up to a period less than seven weeks before L's. death in 1777. In 1772–73 W. wrote from Namur; in 1774 from Danzig, his birthplace, where he spent the months June–September while on leave of absence from his regiment; in 1775–77 from Nimwegen. The letters are written in a very pleasing style and are clearly expressed, numerical examples being frequent. In the two v. (4–5) of the *Briefwechsel*[9] there are references to 20 dated W. letters; we shall later refer to more of the contents of the interesting final letter written from "Helvoetsluys" in 1781 from which we have already quoted (5). The material of v. 5, mainly supplementary to that in v. 4, gives details of computation, often very extended and displaying originality and power, and a number of new tables. He touched on a considerable variety of topics. I shall now refer to only two.

He was interested in the number of primes less than a million[9] (v. 4, p. 497–498, 533; v. 5, p. 380–382, 459) and gave the total as 78461 instead of the correct number 78499. He listed also the totality of numbers under a million having the smallest prime factor p for each p up to 997.

The second topic which interested W. a good deal was the periodicity of rational fractions expressed as decimals[9] (v. 4, p. 523–524, 529, 533–534; v. 5, p. 449–459, 463). In his letter of 1781 W. tells us that already in 1776 he happened on the proof by means of periods of decimal numbers, that the Quadrature of the Circle cannot be expressed by either rational or irrational numbers.[35] He states that he reported his thought in this regard to L. who replied on 8 March 1777, according to W., "Es wird wohl auch—lassen" (apparently an unintelligible extract from an unprinted part of letter[9] XXXVII). L. discovered his proof of the irrationality of π in 1766.

Continuing in his 1781 letter W. reported that finally in 1779, in an investigation of the periods of decimal numbers and their applications to the infinite series of the hyperbola and circle, he had shown that these series cannot be expressed as either rational or irrational numbers,[35] and hence the same results hold for hyperbolic logarithms and the ellipse, which depend upon the hyperbola and circle. W. states that on 28 April 1779 he sent a description of this investigation to Schulze, but received no reply. When Bernoulli later wrote to S. regarding this paper, S. acknowledged its receipt, and stated that use would be made of it when the opportunity presented itself.

Discussion of the periodicity of rational numbers expressed as decimals

seems to have originated with Wallis, as L. mentions. In this connection we have referred to tables of H. Goodwyn [*MTAC*, v. 1, p. 22–23, 67, 100, 372].

7. Wolfram Tables in the Correspondence.—Of the many tables, other than those mentioned above, we shall here refer to only three.

I. In XXV W. sent to L. a table printed on a folding sheet, p. 499–500, indicating 5000 possible "Raketensätze" (rocket mixtures of saltpeter, sulphur, and carbon) for use in artillery firing. As a result of comments by L. in XXV, W. made an entirely new table[9] (v. 5, p. 369–371) for 9402 mixtures, accompanied by discussion of air conditions, angles of fire, calibres of guns, and meal-powder additions, in connection with which latter items a new table is added on p. 377 of v. 5.

II. Levelling tables for France and Rhineland, latitude 50°, the radius of the earth according to La Lande being 3 271 200 toises or 1 692 000 Rhineland Ruthen,[9] v. 5, p. 366–367. These tables are to accompany L's. letter XXIII where he tells W. (p. 491) that the tables were "very well arranged and somewhat similar to tables which he himself introduced while preparing in 1770 a new edition of Picart's paper on hydrostatic balances.

III. In XXXIII W. told L. that he was sending him "something on calculating sines and cosines." This material is in form sort of a table, with part of the calculations,[9] v. 5, p. 443–448. The values of sines and cosines, 18D to 28D, are given for the same 18 angles, namely: 6″, 18″, 54″, 1′, 3′, 5′24″, 6′45″, 9′, 10′, 30′, 54′, 1°7′30″, 1°30′, 1°40′, 5°, 9°, 11°15′, 15°. W. remarks that by means of these results a good number of other values of sines and cosines may be calculated.

In Davisson's correspondence with L. there are enthusiastic references[9] to W., XIII, XIV, v. 4, p. 417–419, 422; and on p. 426 L. writes to D.: "In calculation he possesses an uncommon facility and always observes thereby an admirably planned form." D. remarked, p. 416, that W. had checked his values of sine and cosine for 1″ to 34D.

8. Wolfram's Calculations and the Prussian Academy.—In a brief letter from W. to L. (XXVII), dated Danzig, 29 July 1774, the following sentences occur: "I am now again sending to you a continuation of 320 logarithms, and a third specimen of the same calculation. In such a form have I also since 1750 calculated the common logarithms and I have written out the calculations themselves completely in three volumes, not counting one volume which contains merely preparatory calculations therefor. In the same way I began also two years ago the calculation of hyperbolic logarithms because hitherto I have discovered no other way of calculating *correct* logarithms and preserving them, than by calculating the same in at least two or three ways, and then writing these calculations down in complete form. If this idea appeals to you now, or as you think it over, and if you will indicate a place where one preserves manuscripts of this kind then I shall state in writing that the above volumes after my death shall be transferred without cost to such a place."

At a meeting of the Royal Prussian Academy on 27 August 1774 L. presented the Wolfram plan for consideration, and it was voted that if Wolfram's papers were received as a legacy, they should be carefully preserved in the Academy's archives[9] (see v. 4, p. 511–513, 515–516, 519–521). W. was delighted when L. reported that the Academy approved this idea,

and proceeded to describe three manuscript volumes of his computations (see also v. 4, p. 507–508). In these he had the common logarithms of all primes $<10^4$, to 42D (39D accurate), work which had been done 1752–1761. In 1762 he began the calculation of logarithms of primes <200, to 63D. W. estimated in 1774 that there might eventually be six or seven manuscript volumes. In 1776 W. wrote to L. (XXXII) that he had on 17 June 1775 signed the document setting forth the legacy to the Royal Prussian Academy, and remarked that all manuscript volumes already prepared were carefully marked as part of the legacy, which was to include his factor table of 1743, to 126000 (later enlarged to 300000), and his revision of the Buchner volume (5). W. sent to L. a sample of his factor table,[9] v. 5, p. 438–442, showing that its form was quite different from that of the table in the *Beyträge*.[8] W. stated also that he had notified the "Auditeur militaire," of the Nimwegen garrison, concerning his legacy.

Bernoulli remarks that nothing more is to be found in correspondence concerning this remarkable and praiseworthy legacy. He expresses the hope that the donor will not take it amiss that he had informed the learned world where the fruits of his unremitting industry and rare natural gifts may later be found. From Bernoulli's quoted statement of three years later it seems clear that upon the death of W. there was no disposition of his manuscripts by his executors in accordance with the legacy provisions.

9. Schulze and his Recueil.—J. C. Schulze (1749–1790), was born in Berlin and became a pupil of Lambert during 1770–1772. A Lambert statement concerning his exceptional abilities is contained in the interesting Éloge of Schulze by PAUL ERMAN.[10] He was elected a fellow of the Prussian Academy in 1787 and became a favorite of Frederick the Great. But for the moment we are interested only in Schulze the table-maker, and his notable work, *Recueil de Tables Logarithmiques, Trigonométriques et autres nécessaires dans les Mathématiques Pratiques. Neue und erweiterte Sammlung logarithmischer, trigonometrischer und anderer zum Gebrauch der Mathematik unentbehrlicher Tafeln.* 2 v., Berlin, 1778. The 20 pages of introductory material were in both French and German. In this material Wolfram is referred to on two pages [xii, xvi]. In the last reference we learn that Wolfram had pointed out erroneous results in the trigonometric table of natural sines, tangents, and secants, which Schulze had lifted from the *Opus Palatinum* of RHETICUS. The reason for the difficulty will be apparent from what we have already published [*MTAC*, v. 3, p. 556–557].

But the other page of the Introduction refers to the serious illness of Wolfram when the first volume of the *Recueil* was published. This contained (p. 189–258) Wolfram's extraordinary table of $\ln x$ to 48D. We know from his correspondence with Lambert that he had seen many pages of earlier proof of this table but just when final checking was most desirable, and before he had finished the computation of the values of $\ln x$ for $x = 9769$, 9781, 9787, 9871, 9883, 9907, illness intervened. But Wolfram's computation of the values was published two years later,[8] in 1780. BIERENS DE HAAN thought[17] that possibly this illness had been fatal and that the table was unfinished for this reason.

There is another page of the *Recueil*, v. 1, p. 260, which seems as if it might have been prepared by W. (We know that he prepared p. 188 and most, if not all, of p. 259). On this page are given formulae with coefficients

of the series, for determination of 20D values of hyperbolic logarithms of sines and cosines. All that Schulze stated was (Introd., p. [xiii]) that the formulae were inserted there because of their relation to the preceding table, and because one needed exact logarithms to this extent. There is no reference to these formulae in the W.–L. correspondence.

The way that the W. table came to appear in Schulze's work may be noted. Writing to W. on 30 Nov. 1776 (XXXIV) L. states that having heard of Schulze's work being in the press, he had found, upon consultation with the publisher, that room could be found for a 30D table of hyperbolic logarithms of numbers 1(1)2080, by W. But at this time W. had apparently not only completed his 30D table for all primes and many composites $<10,000$ (1772–1776), but also started his 51D table for this same range—which was later published as a 48D table. It is evident that it was Abraham Sharp's 61D table (which W. finally checked completely by 63D computations,[9] v. 5, p. 462–463) in Sherwin's volume, which got W. interested (1772 +) in extended calculations. For certain ln x, W. made more extended calculations. In[9] v. 5, p. 460 we find (when combined with 48D previously given): 57D for $x = 8191$; 63D for $x = 7499$; 64D for $x = 5827, 6263, 7331, 7487, 9283, 9421$; 65D for $x = 5399, 5471, 5563, 7681$; 66D for $x = 9437$; 71D for $x = 9091$; and 72D for $x = 9973$. Then there is a very curious thing; W. gives 64D to 84D for 12 numbers $x = 193, 199, 251, 503, 521, 643, 997, 1999, 2083, 2699, 4001, 4111$, for which we find in his main table only 48D. I know of no table which gives from the 49th through the 63rd decimals for the last five numbers, or the 62nd and 63rd decimals for the remaining seven. DHL wrote to me, "The 12 numbers are just primes whose simple multiples lie close to numbers whose reciprocals have very simple decimals so that the series which W. used (11) was particularly easy to apply."

10. Pages 188 and 259 of the Recueil.—With reference to p. 188 there are two passages in the correspondence, v. 4, p. 533–534, and v. 5, p. 464. For changing from common to hyperbolic logarithms and conversely, W. computed tables of n/M, and of nM, $n = 1(1)9$ to 66D; but these were abridged to 48D. on p. 188. Next are 44D values of e(42 correct), and of e^{-1} (43 correct); of course $M = \log e$ to 48D is in the table just noted. Then follows a table giving numbers N for which ln N has successively the values 1(1)25, 30, 60. To the value 1, $N = e$(to 27D); to the value 60, N is a 33 figure-6D number. The page inappropriately concludes with 23D values of ln π and log π, values put there by L. to fill an empty space,[9] v. 4, p. 534.

On p. 259, W. (or Schulze) first gives W's. 12 values to 42D of log x, for $x = 9769, 9781, 9787, 9851, 9859, 9871, 9883, 9887$ (only 41D), 9907, 9923, 9967, 10009, that is, for 12 primes. These values seem to have been available to W. since 1761 (8). It is noticeable that these primes include the six that are missing in the ln x published table of 1778 (made good in 1780).[8] Indeed Schulze has told us, 9, p. [xii], that these logarithms were put there because of the blanks on the opposite page, 258. See the comment of KULIK,[12] col. 48. Professor UHLER of Yale University kindly calculated the values of log 9887 and log 10009 to over 60D, and found perfect agreement with W's. results. It turned out that the 42nd and 43rd digits of log 9887, repeated the 40th and 41st.

Why any but the values of logs of the missing primes were here given is not obvious. One would naturally infer that by means of only such values

one might readily compute the missing lns to about 40D by using a table of p. 188.

Already in a letter to L. (XXXI) p. 517, W. remarked that there are 1229 primes <10000 [correct if 1 be not counted as prime], and that given the logarithms of such primes, the logarithms of each of 627549 numbers less than a million might then be found, merely by the addition of two numbers. This statement is elaborated on p. 259, with an auxiliary table.

11. Wolfram's Method of Calculating Hyperbolic Logarithms.—W. has told us that he always used two, and sometimes three, distinct methods for calculating each of his lns (see XXVII, p. 508; XXX, p. 514). We have seen (8) that by 1761 W. had calculated to 42D the common logarithms of all primes, N, less than 10^4, and that in 1762 he began the extension of these calculations of 63D for primes less than[33] 200, which he later continued to $N = 1097$ at least, since he checked the whole of Sharp's table less than 1100 (9). It seems highly probable that a similar extension (or an extension to at least 51D) was calculated for all primes less than 10^4 (note results in 10). We know also that W. had 66D modulus tables (10). Thus one method of making a 51D table of lns (afterwards rounded off to 48D) would be by multiplication of the common logarithms by the modulus reciprocals. But we know that calculation from the following two expansions was basic[9] (XXXI, and v. 5, p. 383–385, 428–437):

$$\ln(1+x) = \sum_{m=1}^{\infty} (-1)^{m+1} m^{-1} x^m$$

$$\ln(1-x)^{-1} = \sum_{m=1}^{\infty} m^{-1} x^m$$

but the expansion for ln $[(1+x)/(1-x)]$ was never used. In finding ln 3343 he gives details of computation. In ln $(1+x)$, x may be taken as $1/234\cdot10^2$, or $1/692\cdot10^3$, or $1/2075\cdot10^4$, or $1/2528\cdot10^6$, or $1/159\cdot10^8$; while in $\ln(1-x)^{-1}$, $x = 1/2651\cdot10^3$, or $1/4611\cdot10^4$, or $1/4806\cdot10^5$. The subsequent details, and dozens of auxiliary tables used in logarithmic calculations in general, are interesting. Here, then, are the three independent methods of calculating the lns.

12. Wolfram's Table and its Accuracy.—De Morgan characterized this table[14] as "one of the most striking additions to the *fundamenta* of the subject which has been made in modern times." Since I know of no wholly correct published statement as to what is in Wolfram's table, I shall now set this forth in some detail. For the most part there are 50 logarithms on a page, arranged in 10 groups of 5. But on 12 pages (238–248, 258), there are 51 logarithms since each of the tenth groups contains 6 entries. Since the table occupies 69 pages we find, therefore, that the values of 48D hyperbolic logarithms are given for 3462 numbers, the last one being the prime number 10 009. [I am assuming that the 6 missing numbers, given by W. two years later, are in place.] These include all values of $x = 1(1)2201$, the 928 following primes and 533 composite odd numbers greater than 2203. Up to 3439 (beyond 2201) the count of composite and prime numbers is exactly the same; but thereafter, primes are the more numerous. In all, there are 1232 primes and 2230 composites in Wolfram's work.

The values of Wolfram's logarithms are each arranged in 8 hexads. I shall now list every reported error, of which I have knowledge, by rewriting the corrected hexads with italicized corrected digits. The number of any particular hexad is indicated by (n), where n is one of the digits 1(1)8; p denotes that the number is a prime:

1. ln 390 (4) 46672*4*
2. ln 829p (3) 294*9*74
3. ln 1087p (5) 0113*45*; (8) 36659*7*
4. ln 1099 (1) 0021*5*5
5. ln 1409p (4) *9*61900
6. ln 1900 (6) 58*1*952
7. ln 1937 (2) 66*3*406
8. ln 1938 (2) 79245*0*
9. ln 2022 (4) 31734*3*
10. ln 2064 (8) 2*80*145
11. ln 2093 (4) *96*5593
12. ln 2173 (8) 22231*1*
13. ln 2174 (8) 50095*7*
14. ln 2175 (8) 18*9*28*3*
15. ln 2194 (8) 0551*1*7
16. ln 3481 (6) 3*9*5248
17. ln 3571p (3) 44844*4*
18. ln 3763 (8) 2796*2*2
19. ln 3967p (5) 791*3*89
20. ln 4033 (8) 4*7*0671
21. ln 4321 (7) *3*50597
22. ln 4357p (8) 4641*8*0
23. ln 4757 (3) *2*81480
24. ln 4891 (4) 76*2*180
25. ln 5123 (6) 37*5*359
26. ln 6343p (2) 1216*3*3
27. ln 7247p (6) 2*5*1021
28. ln 7853p (1) 96*8*650
29. ln 8023 (1) 9900*6*7
30. ln 8837p (3) 00442*3*
31. ln 8963p (6) 3815*3*1
32. ln 9409 (5) 24*3*407
33. ln 9623p (4) 0*5*4318

Thus in the 166176 printed digits in the values of logarithms of the table there are only 38 digits known to be in error. Various considerations, however, suggest that Wolfram's original manuscript may have been almost entirely free from error. While 390 has an incorrect logarithm (no. 1), it is notable that the lns for four of its multiples, and for 195, are correct. So also ln 1658, ln 2174 (except in final digit) and ln 2198 are correct, in spite of the errors in ln 829, ln 1087 and ln 1099 (nos. 2, 3, 4). Hence these facts suggest accuracy of the original manuscript, but bad proofreading perhaps during W's. illness. Three errors (nos. 2, 5, 29) were manifestly caused by the proofreader not recognizing that two inverted 9s should be 6s and an inverted 6 a 9. Similar neglect allowed interchanged numbers to pass in two cases (nos. 3, 11). The four unit-errors in the 48th decimal place (nos. 3, 12–14) are naturally somewhat trivial. Such must suffice by way of justification of our suggestion concerning the most extraordinary accuracy of W's. manuscript. In his letter to Davisson,[9] v. 5, p. 462, W. records "4 Schreib- und Rechnungsfehler"; thus W. himself was willing to confess to a computation error. No one else has, up to the present, noted the error he announced in no. 22.

The first discoverers of 37 of these 38 digit-errors were as follows: BARZELLINI,[8] 1780 (4891); BURCKHARDT,[10a] 1817 (7853); COSENS,[6] 1939 ((8)1087, 2173, 2174, 2175); DUARTE,[24], 1927 (829); DUARTE,[25], 1933 (3967, 8837, 9623); GRAY,[16] 1865 (1409); KULIK,[12] 1824 (390, 1099, 1937, 1938, 2022, 2064, 3481, 3763, 4033, 4321, 4757, 5123, 9409); NYMTP,[28] 1941 (2093, 8023); Peters & Stein, 1922 ((5)1087, 3571) 14; T. M. SIMKISS,[26] 1874 (829, unpublished); STEINHAUSER,[19] 1880 (6343); WOLFRAM,[9] 1781, published 1787 (1900, 4357, 7247, 8963). I have mislaid my reference for the name to be associated with the remaining error, in ln 2194. Can anyone supply this name?

In all three cases (1099, 7853, 8023) where Wolfram has an error in the first hexad, DASE's 7D values are correct in his *Tafel der natürlichen Logarithmen der Zahlen*, 1850.

13. Vega Reprint of Wolfram.—The first reprint of Wolfram's table was in G. VEGA, *Thesaurus Logarithmorum*. Leipzig, 1794, p. 642–684. It will be recalled that W's. 6 correct missing lns and the Barzellini correction of W's.

ln 4891 were published[8] in 1780. Since all errors in W. (12) except, no. 24 in ln 4891, occur in Vega, it would seem as if V. made use of this publication as well as of Schulze's. In that case he made one proofreading slip, or a slip in his checking computation, since for the final digit in the value of ln 9883 he had a 4 instead of the correct 3[.298]. The lns of 25, 520, 7027, were incorrect in the table, but on the two pages of errata the necessary corrections were supplied.

On p. 641 of the Vega volume, the heading of all pages which follow, through 684 is: "Wolframii Tabula Logarithmorum Naturalium." On this page is a somewhat detailed explanation of series to be used for the computation of hyperbolic logarithms. We have seen that such material did not come from Schulze's Wolfram.

Of Vega's table there were four known later editions in 1889, 1896, 1923, 1946; that there was a 1910 edition has not been proved, see *MTAC*, v. 2, p. 163, 283–284. In the 1889 edition all the corrections of the original errata sheets were made in the text throughout. The same is true of the 1923 and 1946 editions. But curiously the original errata sheets are reprinted in the 1923 edition. The very handy miniature edition of 1946 is still in print [*MTAC*, v. 2, p. 161–165]. In the 1896 edition the errors at 1099, 4891, and 6343 have been corrected.

14. Peters & Stein Edition.—Vega's edition of Wolfram's table was the basis of the PETERS & STEIN, Table 13, p. 127–151, in the *Anhang* of 1922. This table contains ln x to 48D, for $x = 2(1)146$ and all later primes less than 10 000. Ten errors of Vega and Wolfram are continued.[1] (829, 1087 last figure, 1409, 3967, 4357 [R.C.A.], 6343, 7247, 8837, 8963, 9623); and also one error of Vega, where W. was correct (9883). No other error has been found. Thus P. & S. were the first to pick up the errors in the fifth hexad of ln 1087, and in the third hexad of ln 3571. Since P. & S. had ln 7853 correct they may have got the result from Kulik,[12] the only prime for which K. gave a correction; Peters would almost certainly have known about this.

15. Callet Extracts.—A few values of Wolfram's table, taken from either the Schulze or the Vega volumes, appeared in F. CALLET, *Tables Portatives de Logarithmes*. Paris, 1795. We here find ln x, to 48D, for $x = 1(1)99$, and for primes to 1097; but Callet adds 48D values for $x = 999980(1)1\,000\,021$; for all but the last six of these numbers Sherwin gave 61D values of the common logs. Callet gives also lns for $x =$ [1(1)1200; 20D], [101000(1)101179; 20D]. Wolfram's errors in ln 829 and ln 1087 are carried over, and also those for ln 829 and ln 1099 in the 20D table.

Of Callet's work there were many reprints or editions.

16. Kulik and Wolfram.—Writing in May 1824 Kulik stated[12] that for two years he had been busy with the revision, extension and publication of Wolfram's 48D table, for all numbers up to 11000. He gave the title of his partially published work as follows: *Canon logarithmorum naturalium in 48 notis decimalibus pro omnibus numeris inter 1 et 11000 denuo in computum vocatus ab* Jac. Phil. Kulik. On January 27, 1825 he reported[12] that the "12th Bogen," presumably through page 192, was being printed (the completed work was to contain 288 pages). He also stated that all printing ought to be finished in the following May, and that the volumes should be on sale by the end of the following September. There is no reference to this work in: (i) any later volume of the *Astronomische Nachrichten*, (ii) any ordinary bibli-

ographies or library catalogues, (iii) even the personal bibliography sent to "Poggendorff" by K. shortly before his death in 1863, nearly 40 years later.

We do know that at Gratz in 1825 Kulik published a xxvi, 286-page factor table—to be found in very few libraries: *Divisores numerorum decies centena millia non excedentium. Accedunt tabulae auxiliares ad calculandos numeri cujuscunque divisores destinatae. Tafeln der einfachern Factoren aller Zahlen unter einer Million nebst Hülfstafeln zur Bestimmung der Factoren jeder grösseren Zahl.*

In the very interesting biography of Kulik (a published portrait is referred to) in C. VON WURZBACH, *Biographisches Lexikon des Kaiserthums Oesterreich*, v. 13, Vienna, 1865, p. 356–359, we find, however, the following entry in a list of Kulik's publications: "*Canon logarithmorum naturalium in notis decimalibus duo de quinquaginta.* (Gratz, 1826), ein logarithmischer Canon mit 48 Decimalen." From this it would appear that this Kulik volume was actually completed and published, but with the title somewhat changed from that of the announcement.

What library has this volume, or any part of it?

17. Thiele and Wolfram.—Elements of mystery are not wholly lacking also in connection with the second attempt to publish an extended edition of Wolfram's great table. Here, at least, four copies are known to exist of: *Tafel|der|Wolfram'schen hyperbolishen|48-stelligen|Logarithmen.|Bearbeitet und erweitert|von*|W. THIELE,|*Herzoglich Anhaltischer Forstmeister a.D.*| Dessau, 1908.|Verlag der Hofbuchdruckerei C. Dünnhaupt.|ii, 108 p. 13 × 20 cm. Bound copies were sold at 6 marks. The volume is clearly printed, but the "Forstmeister" and "Hofbuchdruckerei" combination seems to have prevented the volume from reaching many of the ordinary mathematical publicizing sources. The only references to the work which I could find were in *Jahrbuch ü.d. Fortschritte d. Math.* for 1908, and Deutsche Mathem.—Ver., *Jahresbericht*, 1908. The volume is extraordinarily scarce; the only copies of which I know are at Harvard University (Brown University has a film copy of this), University of Colorado, John Crerar Library, Chicago, and in the private library of C. R. COSENS.[5] I know of no wholly correct published statement as to what is in the table of this volume.

The back of the title page is blank. Then on page 1 the complete introductory text is the following, arranged in fanciful print, diamond-shape, somewhat suggested by the indications for new lines: "Die|natürlichen|oder hyperbolischen|Logarithmen sind in dieser|Tafel bis zu 5000 für jede Zahl| ausgeworfen; von 5000 bis zu 10000|jedoch nur für die dazwischen|liegenden 560 Primzahlen|und für einige andere|ungerade|Zahlen." This statement is wholly correct as far as it goes but there is no reference to the numbers 10001(1) 10014, which are also included in order to fill out 115 pages (2–116), with 50 logarithms on each page. Hence hyperbolic logarithms to 48D are given for 5750 numbers. Thiele took everything from Wolfram's table, that is, the logarithms of 3456 numbers (2230 composites and 1226 primes) the values of six primes being missing (9). Thus Thiele's new work consisted in adding the values of the logarithms of 2288 composite numbers and of 6 prime numbers, in a range where Wolfram had given accurate values for all the 1232 primes less 12 (18). Of the logarithms of these 2288 composite numbers, 2214 are in the first 5000 entries; and therefore beyond 5000, only 74 were added by Thiele. Since copies of Wolfram's table (1946) are still in

print we record just what these 74 numbers are. 5083, 5111, 5267, 5359, 5497, 5543, 5681, 5773, 5899, 5957, 6187, 6233, 6371, 6463, 6593, 6631, 6739, 6859, 6973, 7061, 7199, 7259, 7381, 7511, 7627, 7751, 7859, 7991, 8033, 8113, 8257, 8357, 8479, 8579, 8671, 8723, 8809, 8993, 9089, 9131, 9211, 9367, 9481, 9579, 9683, 9709, 9773, 9889, 9893, 9919, 9937, 9943, 9959, 9979, 9983, 9987, 9989, 9991, 9993, 9997, 9999, and the 13 composite numbers in the range 10000(1) 10014. Thus we can at once tell for what numbers 48D lns may be found in Wolfram (3462), Thiele (5750), Kulik (11000).

There are 562 primes beyond 5000. What about the values furnished by Thiele for the six missing primes? The answer to this question is that the last 6–8 decimal places are incorrect in every one of them. When Cosens wrote his "Note"[5] he had not seen Schulze's work, so that explains part of his following statement: "In the only case I have checked (9883) Thiele is definitely wrong, and I have little doubt that he is wrong in all. Apparently he must have calculated them himself though why he should invariably go wrong at the 40th to 42nd decimal is not clear. I know of no source to 40 decimals, or I should be tempted to suggest that he copied them and made up the last few figures in his head to fill out the line!" How right Cosens is! He did not know of the 42D common logarithms of those six missing primes, which Wolfram had furnished. (8). Hence Thiele simply multiplied these values by $\ln 10 = M^{-1}$ and rounded them off in the manner indicated. Thus Thiele made no computation by series of any new logarithm and, apart from the multiplications here noticed, his other 2288 logarithms seem to have been each obtained by the addition of two numbers, without other checking.

All errors connected with 32 numbers in Wolfram (12) are repeated. Curiously enough Wolfram's error in no. 28(7853) is corrected. This is indeed a major mystery; the only explanation which I can offer is that the typesetter substituted an 8 for a 7, a "mistake" which Thiele did not observe! As a result of these errors Thiele made 14 other errors in connection with the following lns of composite numbers, 8 of these being found by careful calculation of Cosens, and 6 (through additions) by R.C.A.:

1. ln 2487 C (3) $404\underline{6}65$; (8) $45160\underline{3}$
2. ln 2818 C (4) $\underline{3}79132$
3. ln 3261 C (5) $2482\underline{68}$; (8) $92442\underline{0}$
4. ln 3297 C (1) $1007\underline{6}8$; (8) $91826\underline{8}$
5. ln 3800 C (6) $75\underline{8}520$
6. ln 4044 A (4) $73457\underline{6}$
7. ln 4128 A (8) $41\underline{4}505$
8. ln 4145 C (3) $395\underline{3}49$; (8) $2480\underline{49}$
9. ln 4186 A (4) $\underline{38}2825$
10. ln 4227 C (4) $\underline{3}57145$
11. ln 4346 A (8) $35667\underline{1}$
12. ln 4348 A (7) $12\underline{2}908$; (8) $63531\underline{7}$
13. ln 4350 A (8) $3\underline{23}64\underline{3}$
14. ln 4974 C (3) $349\underline{9}75$

But Cosens found also the following three other errors of a different type:

ln 18 (3) $164\underline{69}2$
ln 81 (4) $580\underline{9}80$
ln 3458 (6) $5284\underline{39}$; (7) $\underline{7}55232$

Thus errors have been noted in connection with 55 numbers, 18 of them being among the 2288 whose lns were added by Thiele. In order to state the case fairly it should be noted in final summary that 38 of these 55 errors noted in Thiele's table were due to errors in Wolfram's printed table. A large amount of checking remains to be done. The kind of error for which Thiele is now known to be responsible makes, however, a highly unfavorable impression.

The page and one-half (117–118) following the table is headed "Einige bekannten Constanten," and its most important parts are taken from p. 188 of Wolfram (10): M and $1/M$ to 48D, log π and ln π to 23D, and an abridgement of the table of N, for which $\ln N = 9(1)25$. Among other items is given π to 127D, but there are two errors: *for* 0, *read* 6 in the 108th decimal place; *for* 7 *read* 8 in the 113th.

Following p. 118 is a fly-leaf on the back of which, in a little frame, is:

Hofbuchdruckerei
Weniger & Co.

We now turn to the "mystery" referred to at the beginning of this section. and supplementing the correction mystery. According to Kayser's *Vollständiges Bücher-Lexikon*, v. 36, 1911, and *Hinrichs Halbjahr-Katalog*, v. 221, 1909, the volume we discussed above is referred to as a reprint (Neue Ausg.). In the case of the Harvard University copy, at least, there is no such indication. On the other hand, in the *Halbjahr-Katalog*, v. 217, the details are given for an edition published at Dessau by another printer, "P. Baumann's Nachf.," in 1905; *Bücher-Lexikon*, v. 34, has a less definite reference. This, then, was the original of the work. In no other publication have I been able to find this work mentioned, and scores of inquiries have failed to locate a copy in any library of the world. Such is "mystery" connected not only with Thiele 1905, but also with Kulik 1826. May the publication of this article lead to the discovery of one or both of the volumes.

18. Conclusion.—Such is a summary of the Wolfram information which I have been able to collect. Since some of the books, to which extended reference is made, are not often available in libraries, more details are given than would otherwise be desirable. On the other hand I withstood the temptation to quote passages from Wolfram's letters which definitely suggested a very attractive personality, and a man of high ideals constantly working to the limit of his strength. As a result of this paper I trust that future writers on prominent computers may be able to make concerning Wolfram a more adequate appraisal than was formerly possible. Let us hope that our Dutch friends may succeed in unearthing yet further facts concerning this one of their several outstanding table-makers.

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[1] R. C. A., *MTAC*, v. 1, p. 57; Wolfram, and Peters & Stein errata.

[2] R. C. A., *MTAC*, v. 2, p. 161–163; Wolfram errata.

[3] R. C. A., *MTAC*, v. 2, p. 340, and v. 3, p. 171; Wolfram and Lambert.

[4] R. C. A., "Wolfram's table," *Scripta Mathematica*, v. 4, 1936, p. 99–100, 293.

[5] C. R. Cosens, "A note on the errors in W. Thiele's table of hyperbolic logarithms to 48 decimals, with some remarks on previous tables taken from the original work of Wolfram." Unpublished manuscript dated August 7, 1939 and sent to me in that month. Mr. Cosens (of the Engineering Laboratory, University of Cambridge) computed a number of ln x, to 52D, in testing the accuracy of the table. In writing my paper, some of Mr. Cosens' material has been useful. He had not seen the 1778 edition of Wolfram when he wrote his note.

[6] J. H. Lambert, *Beyträge zum Gebrauche der Mathematik und deren Anwendungen*. Berlin, 3v., 1765, 1770, 1772; v. 3, p. [vi–xi] of the Vorrede contains Wolfram's list of 70 errors in Table I of no. 7 as well as in the Sherwin-Sharp and Clausberg tables. See no. 9(a)(XVIII–XX); v. 2 contains a complete factor table (p. 42–53 + large folding plate.)

[7] J. H. Lambert, *Zusätze zu den logarithmischen und trigonometrischen Tabellen*. Berlin, 1770.

8 J. C. Schulze, "Zusätze und Verbesserungen der Berliner Sammlung trigonometrischer Tafeln," *Berliner Astronomisches Jahrbuch für das Jahr 1783*, Berlin, 1780, p. 191; six missing logarithms, calculated by Wolfram are given, as well as the note by Barzellini, of a Wolfram error. Barzellini's calculation of the values of the 6 missing lns agreed with that of W.

9 *Deutscher gelehrter Briefwechsel Joh. Heinr. Lamberts.* Edited by Johann (III) Bernoulli, Berlin, (a) v. 4, 1784; (b) v. 5, 1785, p. 1–242, and 1787, p. 243–502. (a): Correspondence between Wolfram and Lambert, Letters XVIII–XXXIX, 5 March 1772– 9 August 1777, p. 436–536. XVIII W(L), = letter from W. to L., dated Namur 5 March, 1772 (436–440); XIX L(W) Berlin 21 March, 1772 (441–445); XX W(L) Namur 3 August, 1772 (445–469) (see no. 6, above); XXI L(W) Berlin 19 December, 1772 (469–476); XXII W(L) Namur 8 February, 1773 (477–484); XXIII L(W) Berlin 13 March, 1773 (484–492); XXIV W(L) Namur 5 April, 1773 (493–496); XXV W(L) Namur 26 July, 1773 (497–500); XXVI L(W) Berlin 21 August, 1773 (501–507); XXVII W(L) Danzig 29 July, 1774 (507–509); XXVIII L(W) Berlin 11 August, 1774 (509–512); XXIX L(W) 28 August, 1774 (513); XXX W(L) Danzig 6 September, 1774 (514–516); XXXI W(L) Nimwegen 14 April, 1775 (516–519), notes on L(W) (519); XXXII W(L) Nimwegen 2 April, 1776 (520–522); XXXIII Nimwegen 18 October, 1776 (523–524); XXXIV L(W) Berlin 30 Nov. 1776 (528–529); XXXV W(L) Nimwegen (528–530); XXXVI W(L) Nimwegen 24 January, 1777 (530); XXXVII L(W) Berlin 8 March, 1777 (531–534); XXXVIII W(L) Nimwegen 18 April, 1777 (535); XXXIX L(W) Berlin 9 August, 1777 (536). On p. 531–532 and 536 there are references to another letter W(L) dated 21 March, 1777. In editing these letters Bernoulli supplied annotations. The Nimwegen here evidently corresponds to Mr. Kemperman's Nijmegen. Lambert died in the month following the one in which he wrote his last letter to W.

(b); P. 353–464 constitute supplements to (a), p. 474f, 491, 500, 503, 508, 517, 523, 529. There are many new tables, new Wolfram letters and other valuable material. There are references to 5 other W. letters: to L.(14 October, 1773, p. 368; 30 December, 1773, p. 368, 383; 17 March 1774, p. 392, and for all 3 to v. 4, p. 508); to Schulze (28 April, 1779, p. 463–464), and to Geheim Kriegsrath Davisson in Dantzig (Helvoetsluys, 2 March, 1781, p. 461–463). In connection with this last letter W. records errors in his table at 1900, 4357, 7247, 8963.

10 P. Erman, Académie R. d. Sciences . . . , Berlin, *Mémoires, MDCCXCIV et MDCCXCV*, Berlin, 1799, p. 55–70.

10a J. C. Burckhardt, *Tables des Diviseurs.* Paris, 1817, p. [iii]; correction of ln 7853.

11 J. B. J. Delambre, *Histoire de l'Astronomie Moderne.* V. 1, Paris, 1821, p. 501, 511–513, 519.

12 J. P. Kulik, "Auszug aus einem Briefe . . . 1824, May 13," *Astron. Nachrichten*, v. 3, 1824, cols. 191–192; list of 17 common errors in Wolfram and Vega tables (K. was incorrect in listing an error in ln 1658) and a description of the partial publication of an extension of this table; also *idem*, v. 4, cols. 47–48, April, 1825; further details of this publication. Earlier recordings of two W. errors noted by K. are 4891 (Barzellini), 7853 (Burckhardt), 1900 (Wolfram).

13 C. Gudermann, *J. f. d. reine u. angew. Math.*, v. 9, 1831, p. 362; notes Wolfram error at 1099 (earlier reported by Kulik).

14 A. De Morgan, article "Table," (a) *Penny Cyclopaedia, Suppl.*, v. 2, London, 1846, p. 600–603; (b) *English Cyclopaedia, Arts and Sciences Section*, London, v. 7, 1861, cols. 1000–1001; high praise of Wolfram's work but inaccurate description of its contents.

15 P. Gray, *Tables for the Formation of Logarithms & Antilogarithms, to Twelve Places*, London, 1865, p. 39; Wolfram error in connection with 1409 noted.

16 A. F. D. Wackerbarth, R. A. S., *Mo. Not.*, v. 27, 1867, p. 254; Wolfram error in connection with 1099 noted (already published by Kulik and Gudermann)..

17 D. Bierens de Haan, "Bouwstoffen voor de Geschiedenis der wis- en natuurkundige Wetenschappen in de Nederlanden," Akad. v. Wetenschappen, *Afd. Natuurk., Verslagen*, s. 2, v. 10, 1876, p. 189–191, 202–204; also reprinted in a volume with the above title, 1878, p. 199–201, 212–214.

18 J. W. L. Glaisher, *Report of the Committee on Mathematical Tables.* London, 1873, p. 69, 126. In the Wolfram paragraph on p. 126, line 8, *for* 47 *read* 43; and in the last line, *for* 38, *read* 39.

19 A. Steinhauser, *Hilfstafeln zur präcisen Berechnung zwanzigstelliger Logarithmen.* . . . Vienna, 1880, p. iii, 1; Wolfram error in connection with 6343.

20 D. Bierens de Haan, *Bibliographie Néerlandaise Historique—Scientifique des Ouvrages Importants . . . sur les Sciences Mathématiques et Physiques.* Rome, 1883, p. 309–310.

21 M. Cantor, *Vorlesungen über Geschichte d. Mathematik.* V. 4, Leipzig, 1908, p. 299, 436, 438; notably trivial.

22 R. Mehmke & M. d'Ocagne, *Encycl. d. Mathématiques*, t. 1, v. 4, fasc. 2, 1908, p. 306.

[23] J. HENDERSON, *Bibliotheca Tabularum Mathematicarum*. Part I. Cambridge, 1926; p. 138, 178, three inaccurate statements: (i) about Wolfram's table; (ii) that Gray noted the error discovered by Gudermann (see note 13); (iii) that Wolfram calculated the common logarithm table, p. 259, only once (see note 8); p. 191 more than one misleading statement about the Thiele table.

[24] F. J. DUARTE, *Nouvelles Tables de Log n! à 33 Décimales depuis n = 1 jusqu'à n = 3000*. Geneva and Paris, 1927, p. iii; errors noted in Wolfram's table in connection with 829, 1087, 1409, 1900. On July 23, 1874, T. M. Simkiss reported the 829 case to J. W. L. Glaisher but his result was unpublished before 1928; earlier recordings 1087 (Peters & Stein), 1409 (Gray), 1900 (Wolfram and Kulik).

[25] F. J. DUARTE, *Nouvelles Tables Logarithmiques*. Paris, 1933, p. xxii; eleven Wolfram (1794 table) errors, including 3 of these listed in no. 24—the other 8 being in connection with 3571, 3967, 6343, 7247, 7853, 8837, 8963, 9623—earlier recordings being 3571 (Peters & Stein), 6343 (Steinhauser), 7247 and 8963 (Wolfram), 7853 (Kulik).

[26] NYMTP, *Table of Natural Logarithms*. V. 1–4, Washington, 1941, p. xi, xi, xi, xiii, respectively. W. errors are noted in 829, 1099, 1409, 1937, 1938, 2093, 3571, 4757, 6343, 7853, 8023, 8837, 9623, but the first announcement was made only in connection with 2093 and 8023.

[27] FMR, *Index*. 1946, p. 176–177, 432, 437, 440, 443; inaccurate contents descriptions for Thiele and Wolfram.

[28] J. H. LAMBERT, *Opera Mathematica*, ed. by A. SPEISER. Zürich, v. 1, 1946, p. xiv, xx, 123, 205; v. 2, 1948, p. ix, 70–71. We find in these v. various parts of *Beyträge*,[4] v. 1–3, and the essential parts of the *Zusätze*.[5]

[29] This information was furnished to us through the courtesy of Mr. Eugene Epperson, of Miami University, Oxford, Ohio.

[30] In no bibliography except the *British Museum Catalogue of Printed Books*, and the catalogue of the Royal Observatory Library, Edinburgh, could I find any reference to a Sarganeck: J. J. SCHMIDT, *Biblischer Mathematicus, Oder Erläuterung der Heil. Schrift aus den Mathematischen Wissenschaften . . . Als ein Anhang ist beygefüget Herrn Georg Sarganeck's Versuch einer Anwendung der Mathematic in dem Articul von der Grösse der Sünden-Schulden*. Zullichau, 1736, 27 plates, 11 ff + 672 p. + 16 ff.

[31] A. G. KÄSTNER published 10 volumes beginning with this word, hence it is not easy to determine which one refers to the Leibniz series; perhaps it was *Anfangsgründe der Analysis des Unendlichen*. Leipzig, 1760.

[32] Henderson's statement[23] concerning Sherwin may be recalled here: "No edition of Sherwin was stereotyped and so some of the later editions are less accurate than the earlier. The third edition in 1742, revised by Gardiner, is probably the most correct, although Hutton [Introduction to his *Tables*, p. 40] says it contains many thousands of errors in final figures. With regard to the fifth edition [1770] Hutton remarks, 'It is so erroneously printed that no dependence can be placed in it, being the most inaccurate book of tables I ever knew.' "

[33] The number 200 was undoubtedly suggested to Wolfram by the fact that in his 1726 edition of Sherwin's *Tables*, log 199 was the last entry in Sharp's table as given there.

[34] What I have written here is not very illuminating. Wolfram's complete statement in this regard, however, is as follows (p. 459): "Auf gleichem Grunde habe ich die Cubicwurzeln von Eins bis auf 125, die man in der Artillerie zum Caliberstabe nöthig hat, ohne wirkliche Ausziehung berechnet."

[35] The German passage on which the first of these statements is based is as follows (v. 5, p. 463): "Ich war schon 1776 auf den Einfall gekommen, durch die Perioden der Dezimalzahlen zu beweisen, dass die Quadratur des Zirkels durch keinen endlichen Werth, weder in Rational- noch Irrationalzahlen ausgedrukt werden könne." The second passage is of very similar construction.

RECENT MATHEMATICAL TABLES

794[B, F].—H. E. SALZER, *Table of Powers of Complex Numbers*. NBS, *Applied Math. Series*, no. 8, Govt. Printing Office, Washington, 1950, iv, 44 p. 18 × 26 cm. For sale by Superintendent of Documents, Washington, price 25 cents.

This short table gives the exact real and imaginary parts of $(x + iy)^n$ for $x = 1(1)10$, $y = 1(1)10$, $n = 1(1)25$. The last page gives x^n for $x = 2(1)9$ and $n = 1(1)25$.

The table is unnecessarily repetitive in that it gives powers of both $x + iy$ and $y + ix$. The essential information of the table can be drawn from that

portion which treats $(x + iy)^n$ with $y \leqslant x$. If we write $(x + iy)^n = x_n + iy_n$ then the relations

$$x_{n+1} = x\,x_n - y\,y_n$$
$$y_{n+1} = x\,y_n + y\,x_n$$

were used to construct the table. Perhaps a simpler method would have been to use the fact that x and y are second order linear recurring series,

$$x_{n+1} = 2x\,x_n - (x^2 + y^2)x_{n-1}$$
$$y_{n+1} = 2x\,y_n - (x^2 + y^2)y_{n-1}.$$

The table will be of use for checking formulas involving powers of complex numbers. The numbers x_n, and y_n are examples of LUCAS' functions in the theory of numbers and possess a number of remarkable properties. The table serves the useful purpose of illustrating these properties.

D. H. L.

795[B].—H. S. UHLER, "Table of exact values of high powers of 2," *Scripta Math.*, v. 15, 1949, p. 247–251.

The author has computed, since 1947, a number of isolated powers of 2 of which nine are presented in this note. These are 2^n for $n = 778$, 889,971,1000,2000,2222,3000,3889,4001. The first, third and sixth of these numbers were also calculated by J. W. WRENCH, JR., and the agreement was exact. Fermat's theorem, $2^p - 2$ is divisible by the prime p, was used to check all nine values, although for composite n this required the derivation of "near by" values of 2^p. A simpler and more searching test could have been applied without regard to the character of the exponent n. In fact, if we choose some 10-digit number, quite at random, say 68584 07347 = N, we can find by successive squaring and reduction modulo N that $2^{4001} \equiv 47697\ 23697$ (mod N). This requires less than six minutes with any standard desk calculator. This means that if the author's value of 2^{4001} be divided by 68584 07347 the remainder should be exactly 47697 23697. No doubt it is!

The factors 32009, 224057 of $2^{4001} + 1$ and the factor 24007 of $2^{4001} - 1$, recent results of ALAN L. BROWN, are noted also. The first and last of these factors were used as additional verifications of 2^{4001}.

D. H. L.

796[C].—A. OPLER, "Spectrophotometry in the presence of stray radiation: A table of log $[(100 - k)/(T - k)]$," Optical Soc. Amer., *Jn.*, v. 40, 1950, p. 401–403.

The table mentioned in the title is a 4D table for $T = 2(1)99$, $k = 0(\frac{1}{2})$-5(1)14(2)20 with the obvious restriction that $T > k$. The quantities T and k are "percentages" so that the table is in reality a table of log $[(1 - x)/(y - x)]$ for $x < y < 1$. The table was calculated with IBM tabulator and summary punch. For a slightly larger table see UMT 105.

797[C, O].—C. O. SEGERDAHL, "A table of the interest intensity function for interest intervals of 0.01% from 0% to 7%," *Skand. Aktuarietidskrift*, 1949, p. 15–20.

The table gives in 4 pages 9D values of $\delta = \ln(1 + i)$ for $100i = 0(.01)7$. The table is principally to 8 significant figures and is designed for use in IBM 600 type machines. Great pains were taken to insure the correct last decimal.

The author refers to a previous table of STEFFENSEN[1] which gives δ for $100i = [0(.05)10; 7D]$.

D. H. L.

[1] J. F. STEFFENSEN, "A table of the function $G(x) = x/(1 - e^{-x})$ and its applications to problems in compound interest," *Skand. Aktuarietidskrift*, 1938, p. 47–71.

798[F].—J. W. S. CASSELS, "The rational solutions of the diophantine equation $Y^2 = X^3 - D$," *Acta Math.*, v. 82, 1950, p. 243–273.

If the cubic curve Γ: $Y^2 = X^3 - CX - D$, (C, D given integers) is of genus one, the elliptic arguments of its rational points form an additive group U with a finite number of generators, so that all rational points on Γ may be obtained from a finite number of fundamental points by rational operations (MORDELL, WEIL). Upper limits for W, the number of infinite generators of the group U, have been obtained by BILLING[1] using classical algebraic number theory.

The present author confines himself largely to the equiharmonic case when $C = 0$, but by using deeper results of class-field theory, he is able to delimit more closely the dependence of W on D and the associated real cubic field $R(\delta)$, $\delta^3 = D$.

At the close of the paper, both the fundamental rational points on $Y^2 = X^3 - D$ are tabulated for $|D| \leqq 50$. The class number and fundamental unit of $R(\delta)$ are tabulated for $D = 2(1)50$. In this connection the paper of C. WOLFE[2] cited by Cassels does not tabulate the fundamental unit of $R(\delta)$, but merely a unit $x + y\delta + z\delta^2$ of the ring $R[\delta]$ for $D = 1(1)100$ with x, y, z non-negative.

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[1] G. BILLING, "Beiträge zur arithmetischen Theorie der ebenen kubischen Kurven vom Geschlecht eins," K. Vetenskaps Soc., Upsala, *Nova Acta*, s. 4, v. 11, no. 1, 1938.
[2] C. WOLFE, "On the indeterminate equation $x^3 + Dy^3 + D^2z^3 - 3Dxyz = 1$," Univ. of Calif., *Publ. in Math.* v. 1, no. 16, 1923, p. 359–369.

799[F].—J. LEHNER, "Proof of Ramanujan's partition congruence for the modulus 11^3," Amer. Math. Soc., *Proc.*, v. 1, 1950, p. 172–181.

The congruence referred to in the title is

$$p(1331k + 721) \equiv 0 (\text{mod } 11^3)$$

where $p(n)$ denotes as usual the number of unrestricted partitions of n. The proof is made to depend upon certain modular functions whose Fourier series coefficients are tabulated. The various functions may be described as follows, in which a few liberties are taken with the author's notation:

Let

$$F = \prod_{n=1}^{\infty} (1 - x^n)$$

$$Q = 1 + 240 \sum_{m,n=1}^{\infty} n^3 x^{mn}$$

$$\theta = \sum_{m,n=-\infty}^{\infty} x^{m^2+mn+3n^2}$$

$$\psi = xF(x)F(x^{11})$$

$$G = (121Q(x^{11}) - Q(x))/120$$

$$\Phi = x^5 F(x^{121})/F(x)$$

$$A = \psi^{-1}\theta^2$$

$$B = \psi^{-2}G$$

$$C = (A^2 - 10A - B - 22)/242.$$

The first 23 coefficients in the expansions of

$$\psi^{-1}, \theta^2, A, A^2, \psi^{-2}, G, B$$

are given, reduced modulo $2 \cdot 11^4$. The next 5 or 6 coefficients are given modulo $2 \cdot 11^3$. The first 23 coefficients of C are given modulo 11^2. The next 5 coefficients of C are given modulo 11. The first 30 coefficients of Φ are given modulo 11. The tables may be of use in investigations of the general conjecture of Ramanujan

$$p(n) \equiv 0 (\text{mod } 11^\alpha),$$

where

$$24n - 1 \equiv 0 (\text{mod } 11^\alpha).$$

D. H. L.

800[F].—H. S. Uhler, "A colossal primitive pythagorean triangle," *Amer. Math. Monthly*, v. 57, 1950, p. 331–332.

Exact values are given of

$$a = 2^{4000} + 2^{2001}, \quad b = 3 \cdot 2^{3998} - 2^{2000} - 1,$$
$$c = 5 \cdot 2^{3998} + 2^{2000} + 1.$$

As may be verified, $a^2 + b^2 = c^2$. This pythagorean triangle has almost exactly the same shape as the traditional 3, 4, 5 triangle, the tangent of half the angle A being $\frac{1}{2}(1 + 2^{-1999})$ instead of $\frac{1}{2}$.

D. H. L.

801[G].—Paul Lévy, "Sur quelques classes de permutations," *Compositio Mathematica*, v. 8, 1950, p. 1–48.

The principal results of this work were announced in two notes.[1] The author examines the permutation P_n, among the first n positive integers, definable by (a) $P_n(x) = 2x - 1$ $(2x - 1 \leq n)$, and (b) $P_n(x) = 2(n + 1 - x)$, $(2x - 1 > n)$. The author observes that 1 is invariant, as is $2(n + 1)/3$, if this latter represents an integer. The least common multiple of the order of the cycles of P_n, is the order of the cycle which contains 2. The type of a

cycle is an expression such as $a^{\alpha_1}b^{\beta_1}a^{\alpha_2}b^{\beta_2}\cdots a^{\alpha_\nu}b^{\beta_\nu}$ which defines the succession of operations (a) and (b), which must be performed on x to reobtain this initial element. The order σ is the sum $\sum\alpha_i + \sum\beta_i$. The class of values of n (for which x is an integer) is designated, for given type and σ, by e_σ. It is an arithmetic progression, identified by its least member, n_0. Table I indicates the decomposition of P_n into cycles for $n = 2, 3, \cdots, 45$ and for certain larger values, notably all those for $n \leq 75$ and such that $2n - 1$ is prime, and for those of the form $2^q + \delta$, ($q \leq 11$, $\delta = 0$ or 1). The previous work is generalized: Consider a pack of n cards arranged initially in a certain order, the first being the top of the pack. Place the first on the table, the second under the pack, the next on the table, the next under the pack and so forth alternately, until the pack is reduced to a single card which is placed on the table following the others. The passage from the initial to the final order is the operation Q_n. Table II, (p. 48) gives the decomposition of Q_n into cycles for different values of n. The corresponding indicated order, $\sigma = \Omega(n)$, seems to bear a complicated relation to n, concerning which some partial results are obtained. One notes that $\Omega(127) = 52\,780$, $\Omega(128) = 420$, $\Omega(129) = 8$. In particular if $n = 2^q + 1$, then $\Omega(n) = q + 1$.

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[1] P. Lévy, "Étude d'une classe de permutations," Acad. Sci. Paris, *Comptes Rendus*, v. 227, 1948, p. 422–423; "Étude d'une nouvelle classe de permutations," *ibid.*, p. 578–579.

Editorial Note: The permutations considered above were introduced a decade ago by Narasimha Murti, "On a problem of arrangements," Indian Math. Soc., *Jn.*, new series, v. 4, 1940, p. 39–43.

802[K].—F. J. Anscombe, "Tables of sequential inspection schemes to control fraction defective," R. Stat. Soc., *Jn.*, v. 112A, 1949, p. 180–206.

For special conditions on certain parameters, discussion, examples, and comparison with closely related tables,[1] see Anscombe, above. For background in British work in sequential sampling see Barnard[2] and Anscombe.[3] Barnard's (and Anscombe's) scoring system is, count $+1$ for a good unit, $-b$ for a bad unit, starting score 0, sampling randomly one by one. Accept the batch if score $\geq +H_1$, reject if score $\leq -H_2$. Introduce $b + 1$ ($\equiv$ Wald[4] $1/s$); $R_1 = \dfrac{H_1}{b+1}$ ($\equiv$ Wald $-h_0$), $R_2 = \dfrac{H_2}{b+1}$ ($\equiv$ Wald h_1), p the batch fraction defective, P_p the probability of accepting a batch of fraction defective p, A_p the average sample size for a batch of quality p. In Table I, upper: p (though $p(b + 1)$ is tabled) is given to 5S for $P_p = .99, .90, .50, .10$ .01 for $R_2 = R_1(R_1 = 1(\frac{1}{2})4)$; $R_2 = 2R_1$ for $R_1 = \frac{2}{3}(\frac{1}{3})\frac{5}{3}, 2, \frac{5}{2}$; $R_2 = 3R_1(R_1 = \frac{2}{3}(\frac{1}{3})\frac{5}{3}, 2)$ and three odd pairs of R_1, R_2: (1, 4), (1, 7), (2, 3). In Table II, upper: for each pair of R_1, R_2 above, ratios of p's to 4S for the following ratios of P_p(.99/.90, .99/.50, .99/.10, .99/.01, .90/.10) are given. Table III, upper, gives A_p to 3, 4S (though $A_p/(b + 1)$ is tabled) for each pair of R_1, R_2, for each P_p of Table I upper; also maximum A_p $\left(\text{though maximum } \dfrac{A_p}{b+1} \text{ is tabled}\right)$ for each pair of R_1, R_2 above. Tables I, II, III, lower, give values as above for 19 combinations of $R_1, R_2, K (2 \leq K \leq 12)$ with a truncating condition

(if no decision is reached on inspecting $K(b + 1)$ units, accept the batch if the score > 0, reject if the score < 0). "Rectifying inspection" is defined by. With defective units removed or replaced by good, let N be the batch size, αN the size of first sample, βN the size of each further sample, Y the initial number of defectives in the batch, Z the number of defectives in batch after inspection, ξ the proportion of batch inspected at any stage $(0 < \xi < 1)$, and y the number of defectives so far found $(0 < y < Y)$; then inspection ceases after first sample if 0 defectives are found, after second sample if one defective is found, after $(r + 1)^{st}$ sample if r defectives are found. Required, the maximum probability ϵ that the number of defectives left in the batch be $\geq Z$ (small compared to N).

Table IV: For $\epsilon = .1$, $Z = 5(5)30, 40, 50, 60, 80, 100$ the average sample size A/N is given to 3S for ten values of Y (varying) and for 10 combinations of α and β (varying, each to 4S). Also the AOQL to 2S for each pair (α, β) and the value of Y for which the AOQL is attained.

Table V: is the same as Table IV, for $\epsilon = .01$.

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[1] H. A. FREEMAN, M. FREEDMAN, F. MOSTELLER, & W. A. WALLIS, *Sampling Inspection*, New York, 1948.
[2] G. A. BARNARD, "Sequential tests in industrial statistics," R. Stat. Soc., *Jn.*, Supplement, v. 8, 1946, p. 1–26.
[3] F. J. ANSCOMBE, "Linear sequential rectifying inspection for controlling fraction defective," *ibid.*, p. 216–222.
[4] A. WALD, *Sequential Analysis*, New York, 1947.

803[K].—ALICE A. ASPIN, "Tables for use in comparisons whose accuracy involves two variances, separately estimated," *Biometrika*, v. 36, 1949, p. 290–296.

The tables are designed for use when the precision of a normally distributed estimate, y, of a population parameter, η, depends linearly on two population variances, σ_1^2 and σ_2^2, the sampling variance of y being therefore of the form $(\lambda_1^2\sigma_1^2 + \lambda_2\sigma_2^2)$ where λ_1 and λ_2 are known positive constants. If s_1^2 and s_2^2 are independent estimates of σ_1^2 and σ_2^2, based on f_1 and f_2 degrees of freedom, respectively, then the tables give, for the 5% and 1% probability levels, critical values of the ratio

$$v = (y - \eta)[\lambda_1 s_1^2 + \lambda_2 s_2^2]^{-\frac{1}{2}}$$

to 2D for f_1 and $f_2 = 6, 8, 10, 15, 20, \infty$. These tables can be used in testing the difference between two means of samples from two normal populations whose standard deviations cannot be assumed equal.

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804[K].—P. K. BOSE, "Incomplete probability integral tables connected with Studentised D^2-statistic," Calcutta Stat. Assn. *Bull.*, v. 2, 1949, p. 131–137.

The D^2-statistic is employed as a measure of the distance between two p-variate normal populations. It is a function of the values in samples of

sizes n and n' from the populations, its distribution depending on p, n, n', and the true distance Δ^2 between the populations. The author tables to 3D the upper 5% point of the statistic $C^2D^2/(N + C^2D^2)$ for $\beta = 0$, $p = 1(1)6$, $N = 1(1)50(10)90$ (Table 1), and for $\beta = 5, 20, 50, 100$, $p = 2, 4, 6$, $N = 3(2)49$ (Tables 2–5). Here $\beta = \frac{1}{2}C^2\Delta^2$, $N = n + n'$, $NC^2 = nn'p$. The computations employ recursion formulae previously found by the author.

It is well known that a close relation exists between D^2 and F. In fact, $C^2(N - 1 - p)D^2/Np$ has the distribution of F with p and $N - 1 - p$ degrees of freedom and parameter $\lambda = C^2\Delta^2$. Using this fact, the entries in Table 1 may be obtained at once from readily available tables. The reviewer has checked most of the entries in Table 1 without finding any error. Tables 2–5 seem to be new, adding, for fixed numbers of degrees of freedom, 4 new percentage points to the 8 to 16 points previously given by TANG[1] and the 4 points previously given by EMMA LEHMER.[2]

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[1] P. C. TANG, "The power function of the analysis of variance tests with tables and illustrations of their use," *Stat. Res. Mem.*, v. 2, 1938, p. 126–149.

[2] EMMA LEHMER, "Inverse tables of probabilities of errors of the second kind," *Annals Math. Stat.*, v. 15, 1944, p. 388–398.

805[K].—D. J. FINNEY, "The estimation of the parameters of tolerance distributions," *Biometrika*, v. 36, 1949, p. 239–256.

On page 252 there is a table of weights useful in certain estimation problems. The function tabulated is Z^2/Q, where Z is the ordinate of the normal distribution and Q is the area of the distribution to the right of Z. The table gives 1S or 5D, whichever is greater, for $x = 1.1(.1)9$, x being 5 greater than the argument of the normal distribution. The reviewer recalculated the table, and found no error.

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806[K].—EVELYN FIX, "Tables of noncentral χ^2," Univ. of Calif., *Publ. in Stat.*, v. 1, no. 2, 1949, p. 15–19.

These tables are for the power of certain "chi-square tests" at the 1% and 5% levels of significance. Let $x_1, \cdots, x_f$ denote independent standard normal deviates. For any real $a_1, \cdots, a_f$ the distribution of the non-central chi-square variable

$$\chi'^2_{f,\lambda} = \sum_{i=1}^{f} (x_i + a_i)^2$$

depends only on f and

$$\lambda = \sum_{i=1}^{f} a_i^2.$$

If $\chi^2_f(\alpha)$ is the upper α-point of a central chi-square variable with f degrees of freedom, that is, if

$$Pr\{\chi'^2_{f,0} > \chi^2_f(\alpha)\} = \alpha,$$

then the power of these tests against alternatives characterized by λ is

$$\beta(\lambda) = Pr\{\chi'^2_{f,\lambda} > \chi^2_f(\alpha)\}.$$

The tables give λ as a function of α, β, and f to 3D or 4S for $\alpha = .01, .05$; $\beta = .1(.1).9$, and $f = 1(1)20(2)40(5)60(10)100$. In the table heading on p. 17, $\alpha = .01$ should be changed to $\alpha = .05$, and the opposite change should be made on p. 19.

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807[K].—A. K. GAYEN, "The distribution of Student's t in random samples of any size drawn from non-normal universes," *Biometrika*, v. 36, 1949, p. 353–369.

Unity minus the cumulative density function of t is expressed in the EDGEWORTH series form $P_0(t) + \lambda_3 P_{\lambda_3}(t) - \lambda_4 P_{\lambda_4}(t) + \lambda_3^2 P_{\lambda_3^2}(t)$, where λ's are cumulants of population sampled. Values of the P's are listed on p. 361 (Table 1) for $t = [0(.5)4; 4D]$ and 1(1)6, 8, 12, 24, ∞ degrees of freedom. Four graphs of corresponding probability density function terms appear on p. 362–63.

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808[K].—J. M. HOWELL, "Control chart for largest and smallest values," *Annals Math. Stat.*, v. 20, 1949, p. 305–309.

Given L and S the largest and smallest values in a sample of size n from a normal universe; $\bar{L}$ and $\bar{S}$ their respective means for k such samples; $R = L - S$, $\bar{R} = \bar{L} - \bar{S}$, and $M = \frac{1}{2}(\bar{L} + \bar{S})$. Constants are provided for so-called upper 3-sigma control limits for L above M in the form: U.C.L. $= M + A_3\bar{R} = M + (0.5\bar{R} + 3\sigma_L) = \bar{L} + 3\sigma_L$; and in the form: $a + A_4\sigma_L$ in which $A_3 = 0.5 + 3d_4/d_2$, where $d_4 = \sigma_L = \sigma_S$, $d_2 = E(R)$ for samples of size n from a standard normal universe, $A_4 = d_2/2 + 3d_4$, and a is the mean of the normal universe sampled. Because of symmetry the constants apply to the lower 3-sigma control limit for S in the forms, L.C.L. $= M - A_3\bar{R}$ and $a - A_4\sigma_S$. In Table I values of d_2, d_4, A_2, A_3, A_4 are given for $n = 2(1)10$. (d_2 and $A_2 = 3/(d_2\sqrt{n})$ are available elsewhere.[1] d_4 for $n = 2, 5, 10$ was given first by TIPPETT,[2] and also, for $n = 2(1)10$ by GODWIN,[3] to 7S.) HOWELL's values, given to 3S, are in error in the third figure for $n = 2, 4, 5, 6, 7, 8, 10$. A_3 and A_4 are given to 3S.

In Table II are given values of P_1 and P_2, the power of the conventional 3-sigma control charts for sample range R, and sample mean $\bar{x}$, respectively, for a standard normal universe. Entries are given for $n = 3, 5$, universe mean $a = 0.5, 1.0, 2.0$, and universe standard deviation $\sigma = 1.2, 1.5, 2.0$, where the sizes of the respective 3-sigma regions are determined for $a = 0$, $\sigma = 1$. The power P_3 of the so-called largest and smallest value charts is calculated from $P_3 = Pr(-c < S, L < c)$ where c is determined so that $P_1P_2 = P_3$ for $a = 0$, $\sigma = 1$. Besides P_1P_2, values are also given for N_1,

the smallest integer for which $(P_1P_2)^{N_1} \leq .01$ and for N_2, the smallest integer for which $(P_2)^{N_2} \leq .01$.

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[1] E.g., in *Control Chart Method for Controlling Quality During Production.* Z1–3, Amer. Standards Assn., New York, 1942.

[2] L. H. C. Tippett, "On the extreme individuals and the range of samples taken from a normal population," *Biometrika*, v. 17, 1925, p. 364–387.

[3] H. J. Godwin, "Some low moments of order statistics," *Annals Math. Stat.*, v. 20, 1949, p. 279–285. [See *MTAC*, v. 4, p. 20.]

809[K].—NBS, *Tables of the Binomial Probability Distribution.* NBS Applied Math. Series, no. 6. x + 388. Washington, Govt. Printing Office, 1950. 21.3 × 26.9 cm. Price $2.50.

One of the fundamental distributions in mathematical statistics is the Bernoulli probability function. Let p be the probability of success in a single trial, q the probability of failure, then the probability P_x of exactly x successes in n independent trials, the probability of success being constant from trial to trial, is $\binom{n}{x}p^xq^{n-x}$, and the probability of m or fewer successes is

$$\sum_{x=0}^{m} \binom{n}{x}p^xq^{n-x}.$$

The volume consists of a foreword by Churchill Eisenhart, an introduction (p. v–x) explaining the distribution, scope of the tables, method of preparation, interpolation, applications, and a listing of other tables, mostly unpublished, of the same function. This is followed by two tables, Table 1, (p. 1–195) and Table 2, (p. 197–387). Table 1 gives $\binom{n}{x}p^xq^{n-x}$ for $p = .01(.01).5$, $q = 1 - p$, $n = 2(1)49$, $x = 0(1)n - 1$, to seven decimal places, and Table 2, $\sum_{x=r}^{n} \binom{n}{x}p^xq^{n-x}$, $p = .01(.01).50$, $q = 1 - p$, $n = 2(1)49$, $r = 1(1)n$, to seven decimal places. It is a simple matter to find P_x, $p > .50$, and $\sum_{x=0}^{m} \binom{n}{x}p^xq^{n-x}$ from the results already tabulated. These tables were prepared from tables of the incomplete beta function[1] by Betty Elsen, Amy Norman, and Bonnie Thomas of the personnel of the Department of the Army, and were issued in mimeographed form for limited distribution at the close of World War II. In present form, a photographic reproduction of the mimeographed tables, the preparation of the tables for publication was principally carried out by Lola S. Deming and Celia S. Martin of the Statistical Engineering Laboratory of the NBS. Rarely will difficulties of reading the entries occur. Two instances of such difficulty are the entries for $n = 12$, $r = 6$, $p = .23$, (p. 209) and $n = 30$, $r = 8$, $b = .41$, (p. 272).

For Table 1 an accuracy of ±1 in the seventh decimal place is claimed and for Table 2 an accuracy of ±0.5 in the seventh decimal place. As a spot check the values of $n = 25$, $p = q = .5$ were calculated. P_1 should be .0000007, $P_3 = .0000685$, $P_6 = .0052780$, $P_8 = .0322334$, all within the claimed limits of accuracy. The results for $\sum_{i=x}^{n} \binom{n}{x}p^xq^{n-x}$ are correct as tabulated.

The tables will have many uses. One may mention the operating characteristic function in acceptance sampling and the power function in the testing of hypotheses in statistics. The National Bureau of Standards should take pride in this volume published at such nominal cost. It is desirable that the other tables of the Bernoulli probability function, still unpublished, should see the light of day and that the tables should be extended to high values of n, where the normal curve is a poor approximation in the extreme tails of the distribution.

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[1] K. PEARSON, *Tables of the Incomplete Beta-Function*, Cambridge, 1934.

810[K].—C. R. RAO, "On some problems arising out of discrimination with multiple characters," *Sankhya*, v. 9, 1949, p. 343–366.

The statistic $D_p^2 = \sum_{i,j=1}^{p}\sum s^{ij}d_id_j$, based on p characters, is used to estimate the squared difference between two populations, $\Delta_p^2 = \sum\sum \sigma^{ij}\theta_i\theta_j$. Samples of n_1 and n_2 for each character are drawn from the two populations. The variance-covariance matrix of the sample is s_{ij} with inverse s^{ij}, and the differences between the sample means are indicated by d_i. These are estimates of the population parameters, σ_{ij}, σ^{ij} and θ_i. An example is given for $p = 4$.

The only table given in this article presents the power function for D^2 to 2D when $\phi = 1$, 1.5 and 2, $N = n_1 + n_2 = 16(4)28$, and $p = 1(1)8$, where

$$\phi^2 = n_1n_2\Delta_p^2/N(p + 1).$$

Extensive tables are being prepared of (i) the probability integral of the conditional distribution of R, the statistic used to compare D^2 with p and $(p + q)$ characters

$$R = M_p/M_{p+q}, \text{ where } M_p = 1 + \frac{n_1n_2}{N(n_1 + n_2 - 2)} D_p^2,$$

and (ii) the percentage points of the null distribution of

$$W = \frac{M_{p+q} - M_p}{1 + M_{p+q} - M_p}.$$

It is proposed to compare the relative efficiencies of W and R.

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811[K].—MARJORIE THOMAS, "A generalization of Poisson's binomial limit for use in ecology," *Biometrika*, v. 36, 1949, p. 18–25.

The author introduces a "double Poisson distribution" as follows. Let $X_1, \cdots, X_r$ be independent random variables depending on a Poisson dis-

tribution with parameter λ, and let r itself be a random variable depending on a Poisson distribution with parameter m. Finally let $Z = X_1 + \cdots + X_r + r$ (for example, r may represent the number of clusters, $X_j + 1$ the number of points in the jth cluster). The author studies the distribution

$$Pr\{Z = k\} = \sum_{r=1}^{k} \frac{m^r e^{-m}}{r!} \sum_{\alpha} \prod_{j=1}^{r} \frac{\lambda^{\alpha_j - 1} e^{-\lambda}}{(\alpha_j - 1)!}$$

where $\alpha_1 + \cdots + \alpha_r = k$.

She calculates the mean and the variance and discusses various problems of statistical estimation. These lead to certain elementary equations and a few tables illustrate the practical procedure. Thus Table 5 gives the value of $1 + \lambda$ for given $e^{-m} = .05, .1(.1).9$ and $me^{-m-\lambda} = .05, .1, .2, .3$ to 3D. Other tables of roughly the same size pertain to more complicated functions.

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812[K].—J. W. WHITFIELD, "Intra-class rank correlation," *Biometrika*, v. 36, 1949, p. 463–467.

In analogy with the definition of intra-class correlation for numerical data, it is suggested that an appropriate measure when only ranks are available is the mean of KENDALL's τ coefficient extended over all possible arrangements within classes. In case the classes consist of pairs, the following device affords a compact computation of the mean value. Arrange the pairs as $(a_1, b_1), (a_2, b_2), \cdots, (a_{n/2}, b_{n/2})$, so that each $a_i < b_i$ and $a_1 < a_2 < \cdots < a_{n/2}$. Compute a "score," S, by accumulating the differences for each individual of the numbers of values on his right greater than and less than his own (making no comparisons within pairs). Then, taking $S_p = S - n(n - 2)/4$, the mean value of Kendall's τ is $\tau_p = 4S_p/(n^2 - 2n)$. A table is given for $Pr(S_p \geq S_p')$ to 5D for $n = 6(2)20$ and $S_p' = 0(2)90$ for the case of an uncorrelated universe. S_p is symmetrically distributed about 0 with variance $n(n - 2)(n + 2)/18$. Since $\beta_2 = 3 - 4.32n^{-1}$, a normal test of significance is indicated for large samples.

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813[K].—JOHN WISHART, "Cumulants of multivariate multinomial distributions," *Biometrika*, v. 36, 1949, p. 47–58.

The univariate BERNOULLI and PASCAL multinomial distributions are first considered. Using cumulant distribution functions recurrence relations are obtained from which cumulants to order four are recorded.

Bivariate cumulants to order four are found by recurrence formulae paralleling the univariate case and are also recorded.

Extension to the multivariate case follows from the more simple univariate and bivariate cases. Of importance is the fact that a notation is used which makes the corresponding cumulants of the Bernoulli and the Pascal distributions greatly resemble each other. A complete list of the auxiliary

patterns and the cumulants to the fourth order is given for a particular case ($5 \times 4 \times 3 \times 2$) of the 4-variate multinomial Bernoulli distribution.

There are misprints on p. 52–3.

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814[K].—HERMAN WOLD, *Random Normal Deviates*. Tracts for Computers, No. XXV, Dept. of Statistics, Univ. College, Univ. of London. Cambridge, Cambridge Univ. Press. xiii, 51 p. 16 × 23.2 cm., Price 5 s.

This table contains 25,000 random normal deviates with mean zero, variance one, to 2D. The table was obtained by normalizing the KENDALL-SMITH table of random numbers row by row. Four tests for normality and randomness were applied to each of the 50 pages of the table, to each of the five blocks of 10 pages, and to the entire table. The results showed agreement with normality.

An introduction describes construction of the table and gives techniques for the construction of samples from multivariate normal distributions having prescribed parameters.

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815[L].—A. R. CURTIS, "The velocity of sound in general relativity with a discussion of the problem of the fluid sphere with constant velocity of sound," R. Soc. London, *Proc.*, v. 200A, 1950, p. 248–261.

The functions of Table 1 (p. 261) occur in a static and spherically symmetric metric of space-time. The coefficients e^{ν} and e^{λ} of this metric[1] are derived from a function $V(z)$ where z is a radial coordinate in suitable units. Denoting differentiation with respect to z by dots, $V(z)$ satisfies the nonlinear differential equation

$$3(2\ddot{V} - \dot{V} - V)(1 - V) + (2\dot{V} + 4V - 4e^{z})(2\dot{V} + V - e^{z}) = 0.$$

The pressure is

$$P = \tfrac{1}{12}e^{-z}(2\dot{V} + V) - \tfrac{1}{4}.$$

The table gives 5D values of V and of e^{ν}, together with 4D values of e^{λ} and P for $x = e^{\frac{1}{2}z} = 0(.02).38$, and .38791, the last value corresponding to the first zero of P.

A. E.

[1] A. S. EDDINGTON, *Mathematical Theory of Relativity*. Cambridge University Press, 1923, p. 72.

816[L].—I. FEISTER, "Numerical evaluation of the Fermi beta-distribution function," *Physical Review*, v. 78, 1950, p. 375–377.

The computation of the "Fermi distribution function"

$$f(Z, \eta) = \eta^{2s}e^{\pi y}|\Gamma(s + iy)|^{2},$$

where $s = (1 - \gamma^{2})^{\frac{1}{2}}$, $\gamma = Z/137$, and $y = (\gamma/\eta)(1 + \eta^{2})^{\frac{1}{2}}$ is now in progress at the Computation Laboratory of the NBS. The author gives here 3D

values of f for $Z = 0(10)90$ and $\eta = .6, 1(1)5$, and compares these values with various approximations used in theoretical physics. Percentage errors of the approximations are also given. The so-called non-relativistic approximation is rather poor, the Bethe-Bacher approximation much better, while the Nordheim-Yost approximation rates between the two. The author emphasizes that "The table of Fermi functions, when completed, will in most cases make unnecessary the use of an approximation for the numerical evaluation of $f(Z, \eta)$."

A. E.

817[L].—R. B. DINGLE, "The electrical conductivity of thin wires," R. Soc. London, *Proc.*, v. 201A, 1950, p. 545–560.

The principal quantities occurring in the author's tabulation are

$$\frac{j(r)}{j_0} = \frac{3}{4\pi}\int_0^\pi d\theta \cos^2\theta \sin\theta \times \int_0^{2\pi} d\phi \left[1 - \exp\left\{-\frac{r\sin\phi + (a^2 - r^2\cos^2\phi)^{\frac{1}{2}}}{\lambda\sin\theta}\right\}\right]$$

and

$$\frac{\sigma}{\sigma_0} = \frac{2}{a^2}\int_0^a \frac{j(r)}{j_0} rdr, \qquad k = \frac{2a}{\lambda}.$$

Table 1, p. 553, gives approximate values for very large k of $j(r)/j_0$ to various number of decimal places for $k(1 - r/a) = 0, .2, .5, 1, 2, 5, 10, \infty$.

Table 2, p. 554, gives 3D values of $j(r)/(kj_0)$ for very small (positive) k and $r/a = 0(.25)1$.

Table 3, p. 554, gives 3D values of $j(r)/j_0$ and of σ/σ_0 for $k = .5, 1, 2$ and $r/a = 0(.25)1$.

Table 4 is more intimately connected with the physical problem in hand.

A. E.

818[L].—G. E. FORSYTHE, "Solution of the telegrapher's equation with boundary conditions on only one characteristic," NBS *Jn. of Research*, v. 44, 1950, p. 89–102.

The boundary value problem alluded to in the title presents itself in meteorological theory. Its solution is obtained by means of the GREEN's function

$$G(x, z) = \sum_{n=1}^{\infty} n^{-1} \sin(nx + n^{-1}z).$$

This function is tabulated in the paper to five decimal places for the following 14 values of z: 0, 3.15012, 4.14543, 6.30024, 8.29086, 8.45505, 9.00000, 12.60048, 16.91010, 18.00000, 18.90072, 25.20096, 27.00000, and 36.00000; in each case for $x = -\pi(\pi/36)\pi$. Since there is a discontinuity at $x = 0$, the table gives $G(-0, z)$, $G(0, z)$, and $G(+0, z)$.

The first 18 terms of the series were summed as they stand. The tail was expanded in powers of z, and the first 11 coefficients of this expansion were computed for all x (BERNOULLI polynomials enter in this computation): it is

shown that the remainder then is less than 5 units of the 6th decimal place, and another 5 units were set apart for rounding off errors.

A. E.

819[L].—F. C. FRANK, "Radially symmetric phase growth controlled by diffusion," R. Soc. London, *Proc.*, v. 201A, 1950, p. 586–599.

Table I on p. 589 gives values to varying accuracy of

$$F_n(x) = \int_x^\infty t^{1-n} \exp\left(-\tfrac{1}{4}t^2\right)dt$$

and

$$f_n(x) = \tfrac{1}{2}x^n F_n(x) \exp\left(\tfrac{1}{4}x^2\right)$$

for $n = 3, 2$ and $x = 0(.1).4(.2)4(1)6$. There are also graphs of some related functions. The integrals involved can be expressed in terms of the error function and the exponential integral function.

820[L].—W. HODAPP, "Über die Hermiteschen Funktionen zweiter Art von reellem und rein imaginärem Argument," *Arch. d. Math.*, v. 2, 1949/50, p. 186–191.

Two solutions of the differential equation $y'' - xy' + ny = 0$, $n = 0, 1, 2, \cdots$ are

$$H_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n} e^{-\frac{1}{2}x^2}, \qquad h_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n} \int_0^x e^{-\frac{1}{2}(x^2-u^2)}du.$$

H_n is the HERMITE polynomial, and the author calls h_n the Hermite function of the second kind. (It can be expressed in terms of the parabolic cylinder function.) He gives for h_n convergent expansions in ascending powers of x, explicit forms for $n = 0(1)3$, asymptotic expansions for large x, and indicates briefly some approximations for the real zeros of $h_n(x)$. A numerical table, to 2D, gives upper and lower bounds, approximations, and the exact values of the positive zeros of $h_n(x)$ for $n = 1(1)6$.

A. E.

821[L].—M. KOTANI & H. TAKAHASHI, "Numerical tables of functions useful for the calculation of resonant frequencies of a cavity magnetron," Phys. Soc. Japan, *Jn.*, v. 4, 1949, p. 73–77.

The authors tabulate three functions

$$f(x) = \frac{J_0(x)}{2xJ_0'(x)} + \sum_{m=1}^{\infty} \left\{ \frac{J_m(x)}{xJ_m'(x)} - \frac{1}{m} \right\}$$

$$f(\zeta, \nu; x) = \tfrac{1}{2} \sum_{m=0}^{\infty} \left\{ \frac{\nu J_{\zeta+m\nu}(x)}{xJ'_{\zeta+m\nu}(x)} - \frac{1}{m+1} \right\} + \tfrac{1}{2} \sum_{m=0}^{\infty} \left\{ \frac{\nu J_{\nu-\zeta+m\nu}(x)}{xJ'_{\nu-\zeta+m\nu}(x)} - \frac{1}{m+1} \right\}$$

$$f(0, \nu; x) = \frac{\nu J_0(x)}{2xJ_0'(x)} + \sum_{m=1}^{\infty} \left\{ \frac{\nu J_{m\nu}(x)}{xJ'_{m\nu}(x)} - \frac{1}{m} \right\}$$

$$g(\xi, \nu; x) = \frac{\nu}{2} \sum_{m=-\infty}^{\infty} (-1)^m \frac{J_{|\xi+m\nu|}(x)}{J'_{|\xi+m\nu|}(x)}.$$

The tables are to 4S, mostly, and are as follows:

$f(x)$ for $x = .2(.01)2$
$f(\xi, \nu; x)$ for $x = .2(.1)2.3$ for the following pairs

ν	ξ	ν	ξ
4	0(1)2	8	0(1)4
5	0(1)2	10	0(1)5
6	0(1)3	12	0(1)6

$g(\xi, \nu, x)$ for $x = .2(.1)2.3$ and for
$\nu = 4, \quad \xi = 0, 1$
$\nu = 5, 6 \quad \xi = 0, 1, 2.$

822[L].—A. R. Low, *Normal Elliptic Functions.* Univ. of Toronto Press, Toronto, 1950, 32 p., 15.5 × 23.5 cm, Price $1.25.

Elliptic functions are inversions of integrals whose integrands are rational in the variable of integration and in the square root of a quartic polynomial. Various theories of elliptic functions differ in the standardization which they adopt for the quartic radicand. The most symmetric theory is WEIERSTRASS' where one of the zeros of the quartic is at infinity, and the sum of the other three is zero. The most highly standardized is JACOBI's theory in which two of the zeros are assumed at 1 and -1, while the other two (at $\pm k^{-1}$) are symmetric with respect to the origin. The author's "normalized form" assumes three of the zeros at the standard positions 0, 1, ∞: the fourth zero is called the parameter and denoted by m: it is the k^2 of the Jacobian theory. $m' = 1 - m$ is the complementary parameter, and the case of principal practical importance is $0 < m, m' < 1$.

The standard cubic is $P = p(p - m)(p - 1)$, and it is proved that every elliptic integral can be reduced to a form in which the integrand is a rational function of p and $P^{\frac{1}{2}}$. The elliptic function $p_1 = p_1(u, m)$ is defined by the relation

$$u = \frac{1}{2}\int_{p_1}^{\infty} P^{-\frac{1}{2}}dp$$

and is clearly $\mathrm{ns}^2(u, k)$ of the Jacobian theory. Three other elliptic functions are defined by the relations

$$p_1(u, m)\cdot p_3(u, m) = m, \; p_2(u, m') + p_3(u, m) = 1,$$
$$- p_4(u, m') + p_1(u, m) = 1, \; m' = 1 - m.$$

Values of p_1 to p_4 were computed by the aid of these relations from MILNE-THOMSON's tables of elliptic functions.[1] There are six tables, each to 5D, for $m = 0(.1)1$ and $u/K = .1(.1)1$.

Table I: K, K/K', u. Table II: $\mathrm{sn}(u, k)$. Table III: $p_1(u, m)$. Table IV: $p_2(u, m)$. Table V: $p_3(u, m)$. Table VI: $p_4(u, m)$.

A. E.

[1] L. M. MILNE-THOMSON, *Die elliptischen Funktionen von Jacobi.* Berlin, 1931.

823[L].—R. S. SCORER, "Numerical evaluation of integrals of the form $I = \int_{x_1}^{x_2} f(x)e^{i\phi(x)}dx$ and the tabulation of the function $Gi(z) = \frac{1}{\pi}\int_0^\infty \sin(uz + \frac{1}{3}u^3)du$," *Quart. Jn. Mech. Appl. Math.*, v. 3, 1950, p. 107–112.

Integrals of the form

$$I = \int_{x_1}^{x_2} f(x)e^{i\phi(x)}dx$$

often occur in calculating the wave form due to a source in a dispersive medium, and are frequently evaluated by the method of stationary phase. If the approximation of $\phi(x)$ by its TAYLOR series, in the vicinity of a point of stationary phase, up to and including cubic terms is adequate, the answer can be expressed in terms of AIRY integrals (for which tables already exist[1]) and of the two functions

$$Gi(z) = \frac{1}{\pi}\int_0^\infty \sin(uz + \tfrac{1}{3}u^3)du, \qquad Hi(z) = \frac{1}{\pi}\int_0^\infty \exp(uz - \tfrac{1}{3}u^3)du.$$

$Gi(z)$ and $Hi(-z)$ were computed, on the EDSAC in Cambridge, England, by numerical integration of the differential equations which they satisfy. The computation was performed to 8D, with the interval .02 in z. The tables given in the paper are to 7D for $z = 0(.1)10$: modified second differences are also given. Outside of the tabulated range the asymptotic formulae recorded in the paper are valid to the accuracy contemplated.

In an accompanying note by MILLER & MURSI[2] it is shown how the functions tabulated here together with the Airy integrals can be used to solve the differential equation $y'' - xy = f(x)$ numerically when f is a polynomial.

A. E.

[1] B. A. Math. Tables Committee, Pt.-V.B. *The Airy Integral*. By J. C. P. MILLER, Cambridge Univ. Press, 1946, 56 p.

[2] J. C. P. MILLER and ZAKI MURSI, "Notes on the solution of the equation $y'' - xy = f(x)$," *Quart. Jn. Mech. Appl. Math.*, v. 3, 1950, p. 113–118.

824[L].—E. SAUVENIER-GOFFIN, "Les fonctions $\Gamma(x)$ correspondant, pour les Naines blanches, aux exposants adiabatiques Γ_i des configurations gazeuses," Soc. Roy. Sci., Liége, *Bull.*, 1950, p. 47–54.

4D tables are given of

$$\frac{8x^5}{3(x^2+1)^{\frac{1}{2}}f(x)} \qquad \frac{4x^2+5}{3(x^2+1)} \qquad \text{and} \qquad \frac{8x^5(x^2+1)^{\frac{1}{2}}}{8x^5(x^2+1)^{\frac{1}{2}} - (x^2+2)f(x)}$$

where

$$f(x) = x(2x^2 - 3)(x^2+1)^{\frac{1}{2}} + 3 \operatorname{arc\,sinh} x,$$

for $x = 0(.1)3(.5)10$. Exact values are given for $x = 0, \infty$.

825[L].—DOROTHY A. STRAYHORNE, *A Study of an Elliptic Function*, Thesis, Chicago, 1946. Air Documents Division T-2, AMC, Wright Field, Microfilm No. R c-734 F 15000.

The mathematical part of this paper repeats results which are well known. The numerical part consists of two tables. The first one gives the

numerical values to 4D of the WEIERSTRASS elliptic function $\wp(u)$ for the values of the invariants $g_2 = 37$, $g_3 = -42$ corresponding to the periods $2\omega = 2.2772$ and $2\omega' = 1.3674i$. The argument is imaginary between $.04i$ and $1.36i$ in steps of $.04i$. The second table gives the corresponding JACOBI elliptic functions sn$u$, cn$u$, dn$u$, likewise to 4D, for $u = 0(.05)1$. The value of the modulus is $k = .9585$ corresponding to the choice of the invariants mentioned above.

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826[L].—MARCEL TOURNIER & MARC BASSIÈRE, "Une solution des équations de la couche limite obtenue par la considération de phénomènes transitoires," *La Recherche Aéronautique*, 1948, no. 4, p. 67–72.

On p. 72 are two tables. Table I is of $E_3(z) = \Gamma(\frac{4}{3})\int_0^z e^{-t^3}dt$, $z = [0(.02)1.68; 4D]$, Δ. The values were calculated by the development into powers of z up to $z = 1$. For values of $z > 1$, interpolations were made in the table of PEARSON.[1] A short table of $E_3(.611x)$ ($x = 0(.1)2.8$) is also given to 4D and compared with a "curve of BLASIUS."

R. C. A.

[1] K. PEARSON, *Tables of the Incomplete Γ-Function*. London, 1922.

827[L, P].—GEORG VEDELER, "Basic function for beams with arbitrary constraint," K. Norske Videnskabers Selskab, *Forhandlinger*, v. 22, 1949, p. 171–177.

The author compares a vibrating beam of length l pinned and having fixations f_A, f_B, respectively, at the two ends with a beam whose ends are clamped and subjected to static shears and moments of such a nature that the two beams have the same frequency spectra. The nth deflection mode of the first beam is

$$F_n = A_n(\cosh \alpha_n x - \cos \alpha_n x) - B_n (\sinh \alpha_n x + \sin \alpha_n x).$$

For the case $f_A = f_B = f$, for $n = 1(1)6$ a table of $\alpha_n l$, A_n, and B_n is given for $f = 0(.05).7(.02)1$.

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828[L].—F. G. TRICOMI, "Sul comportamento asintotico dei polinomi di Laguerre," *Ann. di Mat.*, ss. 4, v. 28, 1949, p. 263–289.

A set of four formulae which describe completely the asymptotic behavior of

$$L_n^{(\alpha)}(x) = \sum_{m=0}^{n} \binom{n+\alpha}{n-m} \frac{(-x)^m}{m!}$$

as $n \to \infty$. The four formulae are valid, respectively, in the following four cases: (i) x is in the neighborhood of the origin, (ii) $0 < x < \nu$, (iii) x is in the neighborhood of ν, (iv) $x > \nu$, where $\nu = 4n + 2\alpha + 2$. These formulae are numerically tested on $l_{10}(x) = e^{-\frac{1}{2}x}L_{10}^{(0)}(x)$ for which 4D values are com-

puted both from the exact equation and from the approximation. These values are given together with the absolute error in case (i) for $x = 0(.1)1$, in case (ii) for $x = .5(.5)3(1)6(2)36$, in case (iii) for $x = 34(2)50$, and in case (iv) for $x = 46(2)52$. There is also a table comparing errors of the various formulae in the regions where they overlap.

Asymptotic formulae are also given for the zeros, and they are tested numerically on the zeros of $l_{10}(x)$.

P. 289 is an auxiliary table for the roots of the transcendental equation $x + \sin x = a$, when a and x are expressed in degrees and decimal parts of degrees. If $0 \leq a \leq 90°$, TRICOMI puts $x = \frac{1}{2}a + f(a)$, and if $90° \leq a \leq 180°$ he puts $a_1 = 180° - a$ and $x_1 = 180° - x = 10^{1.4314665}a_1^{\frac{1}{3}} + f_1(a_1)$. 3D tables, with first differences, are given for $f(z)$ and $f_1(z)$ for $z = 0°(1°)90°$.

A. E.

829[L].—CHANG WEI, "Der Spannungszustand in Kreisringschale und ähnlichen Schalen mit Scheitelkreisringen unter drehsymmetrischer Belastung," Nat. Tsing Hua Univ. *Sci. Rep.*, s. A, v. 5, 1949, p. 289–349.

Table I, p. 345, gives the real and imaginary parts of $J_{\frac{1}{3}}(r\sqrt{i})$ and $H_{\frac{1}{3}}^{(1)}(r\sqrt{i})$; Table II, p. 346, the real and imaginary parts of the derivatives with respect to r of the functions tabulated in table I; Table III, p. 347, the real and imaginary parts of $J_{\frac{1}{3}}(-r\sqrt{-i})$ and $H_{\frac{1}{3}}^{(1)}(-r\sqrt{-i})$; and Table IV, p. 348, the real and imaginary parts of the derivatives of the functions tabulated in table III. All four tables are to varying degrees of accuracy, for $r = 0(1)30$. There are several other tables in the paper, but they are of less universal interest.

A. E.

830[M].—W. M. STONE, "A list of generalized Laplace transforms," Iowa State College, *Jn. of Science*, v. 22, 1948, p. 215–225.

This paper presents a table of "generalized Laplace transforms," which is in fact a list of 75 functions $f(s)$, each corresponding to a numerical function $F(k)$, by means of the relation $sf(s) = \sum_{k=0}^{\infty} F(k)s^{-k}$ so that f is essentially the generating function of F. The functions F are chosen from rational functions and combinations of sines and cosines. The functions f are all elementary.

D. H. L.

831[U].—J. C. LIEUWEN, *Kortbestek Tafel*, being v. II of *Zeevaartkundige Tafels uitgegeven op last van het Ministerie van Marine*, The Hague Staatsdrukkerij-en Uitgeverijbedrijf, 1949, ii, 160 p., cloth, 20.7 × 29.5 cm. No price stated.

This collection of navigation tables adds one more to the long list of short methods for the reduction of astronomical sights; it contains many points of interest. Before the main tables there are an arc-time conversion table and four short tables of more or less standard form. Table I is a straight-

forward traverse table giving d'lat and departure to 0'.1 for each degree of bearing and for distances of 10(10)490 and 1(1)9 minutes of arc. Table II is for converting departure into d'long and vice versa and gives the first nine integral multiples of secant (middle latitude) and cosine (middle latitude) for 1°(1°)13°(30')24°(20')34°(10')53°(5')70°(4')72°28'; the full multiplication has to be done by adding the products of successive integers. The curious choice of intervals is evidently dictated by the desire to limit the relative error, without interpolation, to a minimum of 1 in 500. The third table, comprising Tables III a, b, and c, is a collection of altitude correction tables; from these it transpires that

(a) the interpolation table c_1, for latitude, has been "faked" by the incorporation of the second-difference correction on the assumption that the altitude is 70°;
(b) the altitude correction tables have been correspondingly adjusted for a mean value of the latitude difference.

This very considerable complication arises from the curvature of position lines derived from observations at high altitudes; the error due to curvature is precisely that arising from neglect of second differences in interpolations and only those who have striven to find a way of incorporating the corrections in, say, triple-entry tables can fully appreciate the ingenuity of this device. In this case interpolation to the exact D. R. longitude offers no similar difficulties since the interval (1^m) is so small that second differences do not arise; neither do the cross-terms, which are so very difficult to deal with.

Table IV is a collection of small tables for ex-meridian sights; it is ingeniously arranged with one of the arguments in the body of the table in a manner typical of many of the tables to be described in detail later.

The three main tables, A, B, and C, for the calculation of altitude and azimuth from an assumed position are based on a modification of SOUILLAGOUËT'S and DREISONSTOK'S methods, though the table for obtaining the second azimuth angle is new. The astronomical spherical triangle is divided into two right-angled triangles by a perpendicular, length a, from the zenith to the opposite side, meeting this in a point whose declination is K. The first, or time-triangle, is solved directly by double-entry tables. Table A thus gives K to 0'.1, T_1 to 0°.1, (the angle at the zenith, contributing towards the azimuth) and $A = 10^5 \log \sec a$ to the nearest unit, or in some cases the nearest 10 units, for the page heading degrees of latitude (b) from 0° to 71° and with horizontal and vertical arguments hours and minutes of hour angle (P); this table is identical in scope with many others, in particular with Table I of *Hughes' Tables for Sea and Air Navigation*.

The second triangle can be solved in many ways but only by direct double-entry table if the greatest care is taken to avoid difficulties of interpolation and loss of accuracy. Practically all modern tables (with the exception of those of DE AQUINO) solve this triangle by logarithms. Here, however, direct values are tabulated for the second azimuth angle T_2, with the usual (HUGHES-DREISONSTOK) logarithmic solution for the altitude. Table B consists of two parts. On the left-hand pages is given a straight table of $B = 10^5 \log \sec (K \pm d)$ with argument $K \pm d$, where d is the declination, for the range 0°(0'.5)20°(1')89°59'. The interval appears to be determined

by the requirements of the facing page; at intervals of 1′ the half-difference to give the value at the following 0′.5 is also tabulated. The corresponding right-hand pages contain an ingeniously arranged double-entry table for the angle T_2; the horizontal argument is integral degrees of $K \pm d$, covering the same range as on the facing page, and the second argument is $A = \log \sec a$ for which values are given in the body of the table, corresponding to integral degrees of T_2. There is also tabulated T_2, the variation of T_2 with $K \pm d$. The table is based on the formula:

$$\tan T_2 = \tan (K \pm d) \csc a$$

and so is difficult to interpolate when $K \pm d$ is small. Generally, however, the table is satisfactory for its purpose, though the unfamiliar form of entry may confuse users; it is reasonable to ask whether it would not have been better to have accepted the rapidly changing intervals and given T_2 directly with argument A; the present table looks neater but interpolation to tenths of a degree is in places difficult.

Table C is a straight table of $10^5 \log \csc (h - 1°)$ with argument h for the range 0°(0′.5)88°. It is identical in principle with Table II in Hughes' tables, though it does not include the additional decimal for high altitudes. It is entered inversely with argument $A + B$ to give the calculated altitude corresponding to the assumed position.

The second innovation is now introduced in the shape of two tables c_1 and c_2 for the interpolation of the altitude to the D. R. position. The first of these is for interpolation for the minutes of latitude and gives, to 0′.1, the corrections for each minute and for azimuths at intervals of 1°, except within 30° of the meridian when larger intervals are used. This would be apparently a straightforward table of $b \cos$ (Az.), but it is modified in two ways: firstly (as mentioned earlier) by the second-difference correction appropriate to an altitude of 70°, and secondly by the addition of 30′. Thus, all the entries are positive. In using this table the integral degree *nearest* to the D. R. latitude *must* be used; for the exact 30′ the next larger value is the appropriate one. The second table c_2 gives the corresponding interpolation for longitude, but here no second-difference correction is required and it is necessarily triple-entry. The table is in two parts, the one on the left-hand page being essentially a table of the rate of change of altitude with time, $-\cos$ (lat.) $\sin$ (Az.). The horizontal argument is latitude and the body of the table gives the azimuth corresponding to certain approximately equally-spaced values of the rate of change, which are however not specifically given. This page therefore determines a particular line, for which the facing page gives the appropriate multiples (with the addition of 30′) at intervals of 3^m. This is again an ingenious and carefully considered arrangement. It will be noted that the correction is always positive and in fact, is always greater than 15′. Moreover, the indication is that interpolation is always to be done in a forward direction, as opposed to the backward and forward method for the latitude; even so the theoretical error is negligible owing to the smallness of the interval to be covered.

There is also a two-page table and a chart for drawing position lines.

The interpolation tables described are nominally to an accuracy of 0′.1, but this accuracy can only be obtained with some care in use. Direct entry

in the interpolation tables, without mental interpolation or adjustment, will suffice to a reduced standard of 0′.5 in altitude and 0°.5 in azimuth, which the author clearly regards as adequate for most navigational purposes.

Sufficient has been said in the above description of the contents of these tables to show that they have been devised with extraordinary care and skill. The arrangement, layout, printing, paper, binding are all excellent; the figures have heads and tails, are well spaced and are easy to read. The only real criticism is that the ingenuity by which the interpolations to the D. R. position are designed makes it essential to use the tables precisely as instructed. These tables must stand very high among those using the so-called "short-methods."

These tables are the product of prolonged research not only by the author but by a representative committee of Dutch experts. All concerned deserve great credit for an achievement combining ingenuity, practical insight and fine execution. An English edition is contemplated.

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832[V].—Ballistic Research Laboratories, *Tables of Ballistic Functions* $\xi(\theta)$, $c(\xi)$, $s(\xi)$, $X(\beta, b)$, $Y(\beta, b)$, $T(\beta, b)$. Aberdeen Proving Ground, 1949. Approx. 270 leaves, 21.6 × 27.9 cm, tabulated from punched cards.

These are tables of ballistic functions describing the motion of an object subject to the acceleration of gravity and to a resistance proportional to the square of the velocity. The equations describing such motion are solvable by quadrature, as observed by Euler. The approximation (square law drag and constant density) is usable primarily for mortars and for certain rockets, and is now, save for variations of the Siacci method, the only widely used method which avoids numerical integration of the normal equations. These tables give complete trajectories, rather than terminal data only, as given in the tables of Otto and Lardillon.[1] The data have been put in a compact and convenient form and should prove useful.

Table 1 is a repetition of one given by C. Cranz,[1] and corrects a large list of errors (29) occurring in the latter.

The tables were computed on IBM equipment under the direction of I. Schoenberg.

Specifically, the tables comprise:

1. $\xi(\theta) = \int_0^\theta \sec^3 t\,dt$ to (approximately) 7S for $\theta = 0°(1')87°$.
2. $s(\xi) = \sin\theta(\xi)$ for $[\xi = 0(.01)50; 9D]$ where $\theta(\xi)$ is the function inverse to ξ.
3. $c(\xi) = \cos\theta(\xi)$ to 8D for the same range of ξ as in 2.
4. $X(\beta, b) = \int_1^\beta c(b(1 - t))t^{-1}dt$.
5. $Y(\beta, b) = \int_1^\beta s(b(1 - t))t^{-1}dt$.
6. $T(\beta, b) = \int_1^\beta c(b(1 - t))t^{-\frac{1}{2}}dt$.

The last three functions have been tabulated to 8D, for $b = .1(.1)2$ and $\beta = 0(.02)3$, with upper limits of these ranges restricted by $\beta(b - 1) \leq 2$. Save in Table 1, differences through the third order are given. The physical

meaning of the parameters is:

$$b = g/(2kV_s^2),$$

where g is the acceleration due to gravity, k is the usual "resistance" coefficient, V_s is the summital velocity, and $\beta = V_s/(\text{horizontal component of velocity})^2$. Thus, along a single trajectory, b = constant, and the change in horizontal distance, vertical distance, and time is given parametrically with β, as the change in X, Y, and T divided by $2k$, $2k$, and $2kV_s$, respectively.

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[1] C. Cranz, *Lehrbuch der Ballistik*, v. 1, Berlin, 1925.

833[V].—A. van Wijngaarden, "Écoulement potentiel autour d'un corps de révolution," Centre National de la Recherche Scientifique, *Colloques Internationaux*, XIV: *Méthodes de Calcul*, Paris, 1949, p. 72–87.

The paper tabulates functions used in calculating approximate potential flow about a body of revolution with or without angle of attack by a method corresponding to an improvement of von Kármán's source doublet distribution method. The body shape is approximated by a finite number of distributed sources, the ith source distributed along the axis of revolution between $x = (i - 1)a$ and $x = (i + 1)a$. The value of the stream function induced by the ith source at any radius r from the axis and at $x = ka$ is $\psi_i = -Q_ic_{ik}/4\pi$ where Q_i is the source strength and

$$c_{ik} = 2 - \frac{1}{4}\frac{r_k}{a}(\delta_x^2 \tan^2 \tfrac{1}{2}\theta + \delta_x^2 \ln \tan^2 \tfrac{1}{2}\theta).$$

The δ_x^2 indicates the second order central difference in the x direction and $\tan\theta = r_k\{a(k - i)\}^{-1}$. Table 1 (p. 78, 79) gives 4D values of c_{ik} for $r_k/a = 0(.02)2$ and for $k - i = 0(1)9$. The corresponding velocity components are $u_r = (2\pi a^2)^{-1}Q_iu_{rik}$ normal to the axis and $u_x = (2\pi a^2)^{-1}Q_iu_{xik}$ parallel to the axis where $u_{rik} = \frac{1}{2}\delta_x^2 \tan^2 \frac{1}{2}\theta$ and $u_{xik} = \frac{1}{2}\delta_x^2 \ln \tan^2 \frac{1}{2}\theta$. The u_{xik} and u_{rik} are given in Tables 2 and 3, respectively (p. 80–83), for the same range of parameters. The velocities induced by similarly distributed doublets are needed when the body moves other than parallel to its own axis of revolution. If ϕ is the angle about the axis of revolution, the axial velocity induced by a doublet of moment M_i is $(4\pi a^3)^{-1}M_iu'_{xik}\cos\theta$ while the radial velocity is $(4\pi a^3)^{-1}M_iu'_{rik}\cos\phi$, where

$$u'_{xik} = ar^{-1}\delta_x^2 \cos\theta; \quad u'_{rik} = ar^{-1}\delta_x^2 (\cot\theta\cos\theta).$$

These two functions are given in Tables 4 and 5, respectively (p. 84–87), for the range of parameters given above.

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MATHEMATICAL TABLES—ERRATA

In this issue references have been made to Errata in RMT **806** (Fix), **808** (Howell), **832** (BRL). See also p. 194, 197, 198.

174.—R. L. ANDERSON & E. E. HOUSEMAN, *Tables of Orthogonal Polynomial Values Extended to* $N = 104$. [*MTAC*, v. 1, p. 148–150].

On p. 669, $n = 101$, col. 4, argument 23

for 26593 *read* 26592.

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175.—L. J. COMRIE, *Chambers's Six-Figure Mathematical Tables.* [*MTAC*, v. 3, p. 86–87.]

In v. 1, table VII, p. 499, line 1,

for Sh and Th *read* −Sh and −Th, respectively.

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176.—L. E. DICKSON, "Finiteness of the odd perfect and primitive abundant numbers with n distinct prime factors," *Amer. Jn. Math.*, v. 35, 1913, p. 413–422.

A complete recalculation of the list of primitive abundant numbers p. 420–422 shows the following errata.

Delete:	$3^2 \cdot 5 \cdot 11^2 \cdot 19^2$,	$3^2 \cdot 5 \cdot 11^3 \cdot 19$,	$3^6 \cdot 5^6 \cdot 19 \cdot 73^2$,	$3 \cdot 5^2 \cdot 7^4 \cdot 29$
Insert:	$3 \cdot 5^4 \cdot 7^2 \cdot 31$,	$3^2 \cdot 5 \cdot 11^2 \cdot 19$,	$3^3 \cdot 5^5 \cdot 17^3 \cdot 61^2$,	$3^4 \cdot 5^4 \cdot 19 \cdot 53$
	$3^4 \cdot 5^4 \cdot 19^2 \cdot 61$,	$3^5 \cdot 5^5 \cdot 19^3 \cdot 83$,	$3^6 \cdot 5^2 \cdot 19^3 \cdot 53^2$,	
	$3^6 \cdot 5^4 \cdot 19 \cdot 71^2$,	$3^6 \cdot 5^5 \cdot 19 \cdot 73^2$,	$3^6 \cdot 5^7 \cdot 17 \cdot 127$,	
	$3^7 \cdot 5^2 \cdot 19^3 \cdot 53$,	$3^7 \cdot 5^4 \cdot 19 \cdot 73^2$,	$3^7 \cdot 5^7 \cdot 17^2 \cdot 233$.	

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177.—C. F. GAUSS, "Tafel zur Verwandlung gemeiner Brüche mit Nennern aus dem ersten Tausend in Decimalbrüche," *Werke*, v. 2, Göttingen, 1863, 2nd ed., 1876, p. 412–434.

GLAISHER[1] stated that he had compared this table of decimal periods with GOODWYN'S *Table of Circles*[2] and found the latter to be more accurate. Apparently, Glaisher never published a list of discrepancies in the two tables.

The following 22 errata have been found in Gauss' tables as the result of a complete recalculation of his data.

Two typographical errata exist in the designation of the periods. The period associated with 47 should be designated (0), not (1). The second period shown in connection with 243 should be marked (2), not (3).

Prime	Designation of period	*for*	*read*
59	(0)	2472881355	2372881355
233	(0)	7959914163	7939914163
		2789799570	2789699570
271	(52)	23447	23247
331	(0)	2779466193	2779456193
359	(1)	1058485821	1058495821
397	(0)	303022670	403022670
419	(0)	1183317422	1193317422
443	(1)	5869574492	5869074492
541		5101663385	5101663585
587		1763202725	6763202725
653		4211322312	4211332312
719		1390320584	1390820584
773		6921096675	6921086675
863		1657010438	1657010428
883		1925754813	1925254813
		6602441506	6602491506
967		7269966928	7269906928
977		9979529178	9979529170
983		0315361159	0315361139
		3550556052	3550356052
991		9845610494	2845610494

At my request Professor R. C. ARCHIBALD has compared the preceding data with the corresponding results in Goodwyn's table. He reports that these errata in Gauss' table do not coincide with any of the known errata in Goodwyn's work.

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[1] J. W. L. GLAISHER, "On circulating decimals," Cambridge Phil. Soc., *Proc.*, v. 3, 1877, p. 185–206.
[2] H. GOODWYN, *A Table of the Circles*, etc., London, 1823 [*MTAC*, v. 1, p. 22–23].

178.—M. KRAITCHIK, *Recherches sur la Théorie des Nombres*, v. 1, Paris, 1924.

In Table IV, p. 229, $N = 2273$, $\rho = 97$

for 386 *read* 381

For other errata in this table see *MTAC*, v. 3, p. 372, MTE **147**.

D. H. L.

UNPUBLISHED MATHEMATICAL TABLES

105[C].—A. OPLER, *Table of* $\log [(1 - x)/(y - x)]$. Tabulated from punch cards and deposited in UMT File.

This is a 5D table for $x = .02(.01).99$, $y = 0(.005).05(.01).2$ $(y > x)$. It is a slightly more elaborate table than the one reported in RMT **796**.

106[F].—A. S. ANEMA, *Table of the number of primitive right triangles with perimeters not exceeding* $2N$. Manuscript on deposit in UMT FILE.

Let $T(N)$ denote the number of primitive pythagorean triangles whose semi-perimeters do not exceed N. Then it is known[1] that

$$T(N) = \pi^{-2}N \ln 4 + O(N^{\frac{1}{2}} \log N). \tag{1}$$

The present table gives $T(N)$ for $N = 500(500)60000$ and is based on actual lists of pythagorean triangles compiled by the author. This table extends considerably one given[1] by D. H. L. for $N = 500(500)5000$. The table exhibits the remarkable smallness of the error term in (1). At $N = 60000$ we find, for instance, that

$$T(N) = 8430,$$

while

$$\pi^{-2}N \ln 4 = 8427.659.$$

[1] D. H. LEHMER, "A conjecture of Krishnaswami," Amer. Math. Soc., *Bull.*, v. 54, 1948, p. 1185–1190.

107[F].—A. S. ANEMA & F. L. MIKSA, *Tables of primitive pythagorean triangles with equal perimeters.* Typewritten manuscript (21 p.) on deposit in UMT File.

The table lists 182 sets of primitive pythagorean triangles (A, B, C) in sets of 3 (or 4) which have equal perimeters less than 10^6 together with the "generators" of each triangle. There are seven sets of 4 such triangles, the smallest one being

(86099, 99660, 131701)
(133419, 43660, 140381)
(151811, 13260, 152389)
(9435, 153868, 154157)

All 4 triangles have the same perimeter $317460 = 2^2 \cdot 3 \cdot 5 \cdot 11 \cdot 13 \cdot 37$.

108[F].—A. GLODEN, *Table de factorisation des nombres* $N^4 + 1$ *dans l'intervalle* 3001–6000. Manuscript deposited in UMT File.

This typewritten table of 28 leaves gives data on the factors of $N^4 + 1$ for $N = 3001(1)6000$. For the majority of N's the complete factorization is given. In some cases the factorization is incomplete or even entirely unknown. Any unknown factor exceeds 600000.

This table extends the author's previous table for $N = 1001(1)3000$ and the table of CUNNINGHAM for $N = 1(1)1000$ [*MTAC*, v. 2, p. 211; v. 3, p. 118–9].

109[I, K].—W. F. BROWN JR. & C. W. DEMPSEY, *Tables of Orthogonal Polynomials and of their Derivatives.* Photostat, 34 leaves, deposited in UMT FILE.

The polynomials tabulated are those of CHEBYSHEV and GRAM, used for curve fitting, and are what FISHER & YATES denote by $\xi_r'(x)$ [see *MTAC*, v. 1, p. 148–150, FMR, *Index*, §23.82]. The work is in three parts.

Table I gives the coefficients of $\xi_r'(x)$ for n points for all $r < n$ and for $n = 3(2)25$, as well as

$$S_r = \sum_{|k| < \frac{1}{2}n} \{\xi_r'(k)\}^2.$$

Table II gives all the derivatives of the polynomials in Table I as far as $n = 15$.

Table III gives a table of three integer parameters to enable the user to pass easily from the polynomials in Table I to the corresponding polynomials T_r of AITKEN.

AUTOMATIC COMPUTING MACHINERY

Edited by the Staff of the Machine Development Laboratory of the National Bureau of Standards. Correspondence regarding the Section should be directed to Dr. E. W. CANNON, 225 Far West Building, National Bureau of Standards, Washington 25, D. C.

TECHNICAL DEVELOPMENTS

Report on the Machine of the Institut Blaise Pascal

1. The fundamental characteristics of the machine being built for the Institut Blaise Pascal are as follows.

a) It will be a laboratory machine, of which the elements can be changed or increased in number without upsetting the general structure of the machine.

b) It will be a parallel machine.

This last characteristic has led to the study of calculating devices first, for we assumed from the beginning and still think that the problems of memory and control cannot be solved a priori: their solutions depend upon the characteristics of the calculating devices and upon the nature of the problems to be attacked.

2. Mathematical investigation has led to a method[1] of performing division and square rooting in the binary system, reducing these operations to a series of additions and of subtractions of the same duration as the series which constitutes multiplication—lasting some microseconds only.

In consequence of this result:

a) the arithmetic unit is devised to perform automatically the basic operations of addition, subtraction, multiplication, division, and square rooting, but

b) the only operations actually performed in the calculating organ are addition and subtraction (and repeated sequences of these).

3. The computer built on these principles is composed of:

a) three accumulators, M, X, and P, where are stored or are formed: in M, the multiplicand and the divisor; in X, the multiplier, the quotient, and the square root; and in P, the product, radicand, and the dividend, and

b) of a subroutines program which controls the sequence of additions and subtractions of which are built up the basic arithmetic operations.

Furthermore, the quotient and the square root are transferred to P at the end of these operations, in order that the result of the operation shall always

be read in the same register; the relations of the arithmetic organ to the other parts of the machine are thereby simplified.

4. Each accumulator is formed of standard binary elements of which Figure 1 gives the circuit. The left-hand double triode serves as a register. Carry is effected by the delay line *ES*. The two triodes of the right-hand tube, *L*, act independently; the first, L_c, reshapes the register impulse coming from 7 or from *S* before it enters the register triode, and the second, L_s, transfers the digit held by this binary element to another binary element. Indeed, a positive pulse coming to the grid of this triode from 14 will emerge from 9 to enter a binary element of another accumulator only if the potential of the grid of L_s has already been statically raised, which occurs when the triode F_v holds a 1.

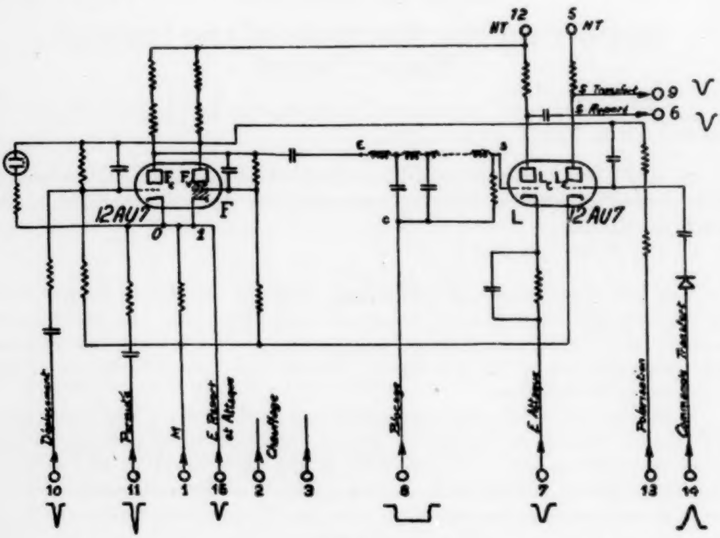

Fig. 1

The shifting of a number one position to the left in an accumulator is performed by introducing a negative pulse at 10. If the triode F_c indicates 0, nothing is changed; if not, it is changed from 1 to 0. This is equivalent to adding the digit to itself, that is to say, multiplying by two which is identical with shifting one position to the left.

Clearing is accomplished by the same means as shifting, except that the carries are simultaneously suppressed.

"Permutation," accomplished by a negative pulse at 11, changes all 1's to 0's and all 0's to 1's. The addition of the number so obtained, combined with an end-around-carry, is equivalent to subtracting the original number. This manner of realizing subtraction is more simple than adding the true complement of the subtrahend.

Figure 2 shows the group of accumulators M, X, and P, of the reduced model, of which the binary capacity is $8 \times 8 \times 16$ while the capacity of the complete model will be $48 \times 48 \times 96$ which corresponds to $15 \times 15 \times 30$ in the decimal system.[2]

5. Internal programs, e.g., for multiplication, are controlled by permanently-wired arrangements of delay tubes and pulse generators. These, once stimulated, carry out automatically the sequences of addition and shifting required for the operations in question. The permanently-connected arrangements have the advantages of permitting the adjustment of each element separately and of reducing to the indispensable minimum the duration of transmission and execution of the successive orders; this duration is reduced, indeed, for each elementary operation of transfer, permutation, etc., to the excitation time of one tube, that is, to less than a microsecond.

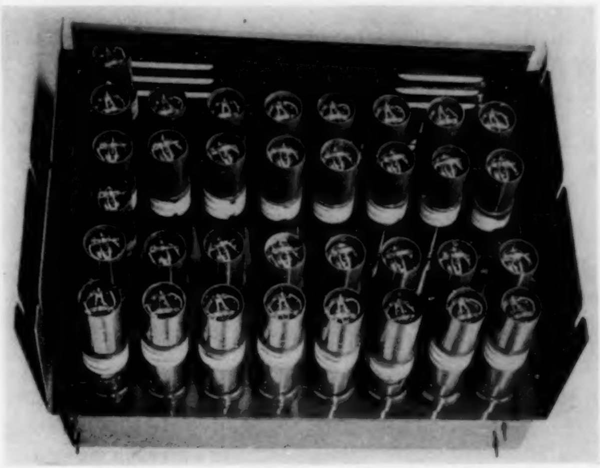

Fig. 2

6. The reduced model of the computer actually constructed has an internal memory composed, for each register, of a single gas diode (neon lamp) in each binary position. Each diode is doubly driven, by a "row" tube which controls the diodes in which are to be written the digits of a particular number to be put in the memory, and by a "column" tube which controls the diodes affected by a particular order of binary numeration. The appearance of the model actually constructed is shown[3] in Figure 3.

This memory device, which already has functioned correctly for more than 300 hours, will be adopted for the large machine, for it has, besides the advantages of ruggedness and simplicity, that of permitting the transfer of a number from the memory to the calculator in less than 2 microseconds; it thus appears to be one of the fastest memories existing.

This internal memory will be limited to the number of registers needed to carry out the usual sequences of intermediate calculations. It will be

built on panels of about twenty registers; new panels will be added if and when the study of particular problems shows their usefulness.

7. A memory for constants, analogous to the diode memory except that the diodes are replaced by resistors, will hold the usual constants: π, M, $180/\pi$, etc., and the coefficients of the formulas for approximating the usual functions $\log x$, $\sin x$, $\tan x$, e^x, etc.

These formulas are, mainly, the expressions of transcendental functions in polynomial form, of which a simple example is a development in series. However the facility with which our machine can perform division and square rooting permits the use of more complicated approximation formulas, which reduces the time required to calculate a function while still retaining simplicity in the internal subprogram. The discovery of the best of these formulas for each given function is a mathematical problem posed by the

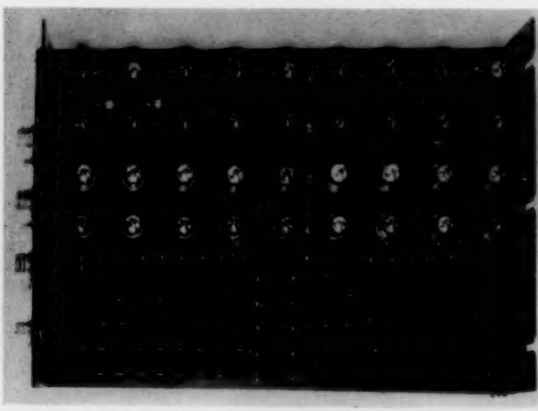

FIG. 3

structure of the machine. To each of these formulas will be attached a subprogram of the same kind as that which has been described in section **5**. For almost all of the functions, even those given by experimental curves, we expect to be able to replace function tables by approximation formulas which will, on the average, give the result more rapidly.

8. The general arrangement is such as to simplify coding as much as possible. To this end we have applied the principle used by Professor AIKEN in Mark I. For each operation the programmer has to write one, two, or three numbers, which designate the operation to be performed, the register to which it is to be applied, and the register which furnishes the operands. For example:

a) division—one number, which indicates the operation of division;

b) clearing—two numbers, that of the operation of clearing and that designating the register to be cleared;

c) transfer—three numbers, that of the operation of transfer, that of the register receiving the number transferred, and that of the register yielding the number.

Each order is therefore represented as a binary number. This number is read by a photocell from a variable-density film. The test model, with the film mounted on a disk, is shown in the frontispiece. The orders are entered in series on the film and are conveyed in parallel to the tubes seen at the top of the device for later transference in parallel to the several parts of the machine. In the final machine the order film may be of great length. Its speed will be of the order of three meters per second.

9. For the external memory and the output mechanism no decisions have yet been made. It is probable that several devices, such as magnetic tapes, photographic films, electroscription, etc., will be used together.

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[1] L. Couffignal, "Calcul d'un quotient ou d'une racine carrée dans le systéme de numération binaire," Acad. des Sci. Paris, *Comptes Rendus*, v. 229, 1949, p. 488–489.

[2] In the accumulator assembly the two tubes of each stage, the accompanying circuit elements and the neon light for reading the stored digit are all mounted inside a heavy cardboard tube which serves at the same time to protect the elements and to support the delay line. Each element is supplied with a special nylon mounting.

[3] The figure shows at the bottom of the panel the tips of the neon diodes in 10 rows of 9 each. These correspond to 10 numbers of 8 digits and a sign. At the top are the selection tubes for the numbers and for the binary positions.

The Operating Characteristics of the SEAC

In a previous report [*MTAC*, v. 4, p. 164–168] the logical design of the SEAC was described in detail. The following gives some account of the machine's operating characteristics.

(1) *Basic repetition rate*—1 megacycle per second.

(2) *Type of number representation*—binary system, serial.

(3) *Word length*—number and instruction words consisting of 45 binary digits (44 numerical digits and algebraic sign), equivalent in precision to approximately 13 decimal digits.

(4) *Instruction systems*—two modes of operation are available, namely:

(a) *4-address system*—typical instruction word specifying 10-digit addresses of (α) first operand, (β) second operand, (γ) result of operation, and (δ) next instruction

(b) *3-address system*[1]—typical instruction word specifying 12-digit addresses of (α) first operand, (β) second operand, and (γ) result of operation with instructions normally arranged in consecutively-numbered memory locations.

Before starting computation, the computer is set for operation in the particular instruction system desired.

(5) *Types of internal memory*

(a) *Serial*—512 words stored in 64 mercury acoustic delay lines containing 8 words each with access times as follows: maximum, 336 microseconds; average, 168 microseconds

(b) *Parallel*[1]—512 words stored in 45 electrostatic (Williams) tubes

holding 512 binary digits each with access time as follows: average for typical operation, 12 microseconds

(c) *Serio-parallel*—32 words stored in 3 electrostatic (Williams) tubes holding 512 binary digits each with access time as follows: average, 1728 microseconds (experimental system).

The serial memory can be used in conjunction with either of the other two types. The experimental serio-parallel type will be replaced by the 45-tube fully-parallel system as soon as construction of the latter is completed. Pending this, the 3-tube system will be used for evaluating comparative performance under practical operating conditions of various types of memory tubes, e.g., Williams tube, selectron tube, etc.

Provision is made for possible increase of the combined memory capacity up to 4096 words.

(6) *Basic operations and performance times*—

Operation[2] (with abbreviation)	Time (in milliseconds) for complete operation (including access time) Serial acoustic memory Max.	Min.	Av.	Parallel electrostatic memory
1. Addition (*A*)	1.5	0.2	0.9	0.2
2. Subtraction (*S*)	1.5	0.2	0.9	0.2
3. Multiplication	3.6	2.3	3.0	2.4
(a) Major part, unrounded (*M*)				
(b) Major part, rounded (*R*)				
(c) Minor part (*N*)				
4. Division (*D*)	3.6	2.3	3.0	2.4
5. Comparison (A conditional transfer of control based on value of arithmetical result)	1.2	0.2	0.7	0.2
(a) Algebraic value (*C*)				
(b) Absolute value (*K*)				
6. Logical Transfer (*L*) (An arbitrary partial word transfer for the purpose of forming composite words)	1.5	0.2	0.9	0.2
7. Input-Output Control			50 (See footnote 3)	
(a) Read-in (*T*)				
(b) Print-out (*P*)				
(c) Reverse motion (*7*)				

(7) *Number of components* (approximate quantities)—

	Tubes	Germanium diodes
Serial memory (512 words)	400	3,500
Parallel memory (512 words)	300	4,500
Computer exclusive of memory	350	7,500
Totals	1,050	15,500

(8) *Power requirements*—15 kw.

(9) *Net floor space*—150 square feet.

ELECTRONIC LABORATORY STAFF

NBS

[1] The 3-address system and the parallel memory are still under construction.

[2] In the initial model, shift effects may be obtained by means of multiplication, division, or addition. Provision is made for the possible later addition of a special shift instruction, as well as other additional instructions.

[3] This is true for the initial single channel magnetic wire with 8-word block. The interim system uses modified teletype equipment. Provision is made for the addition of other types of serial and/or parallel input-output equipment.

Discussions

Binary—Decimal Conversion on a Desk Calculator

The EDVAC normally employs 43-binary-digit numbers less than one in absolute value. The machine will generally be programmed so as to convert automatically its input data to this form and its numerical results to decimal form.[1] We describe here procedures for performing these conversions upon a 10-place desk calculator. These procedures are designed for the use of problem preparers, coders, and maintenance personnel for the occasional conversion of decimal numbers to 43-binary-digit numbers and conversely.

A 43-binary-digit number furnishes, roughly, the same degree of approximation as a decimal number of 13 digits. This degree of approximation is maintained in the methods here described, even though only a 10-place desk calculator is used, by taking advantage of the fact that $1/8^j$ for $j = 1, 2, \cdots, 15$, can be approximated to 14 decimals by a 10-decimal-digit number multiplied by an appropriate power of 10. The processes may be terminated at any point if full accuracy is not required. The resultant binary or decimal equivalent, as the case may be, is recorded automatically on one of the two dial registers of the desk calculator.

The binary number, $+.c_1c_2 \cdots c_{43}$, ($c_j = 0$ or 1), is an abbreviation for the quantity

$$N = c_1 2^{-1} + c_2 2^{-2} + \cdots + c_{43} 2^{-43}.$$

This quantity may be expressed in the form:

$$\begin{aligned} N &= (4c_1 + 2c_2 + c_3)8^{-1} + (4c_4 + 2c_5 + c_6)8^{-2} + \cdots \\ &\qquad + (4c_{40} + 2c_{41} + c_{42})8^{-14} + 4c_{43}8^{-15} \\ &= d_1 8^{-1} + d_2 8^{-2} + \cdots + d_{14} 8^{-14} + d_{15} 8^{-15}, \end{aligned}$$

(where $d_k = 0, 1, 2, 3, 4, 5, 6, 7$ for $k = 1, 2, \cdots, 14$; and where $d_{15} = 0$ or 4) which we write as $+.d_1d_2 \cdots d_{15}$, an octal number.

We convert from binary to octal by arranging the binary digits in groups of threes starting from the binary point and converting each group to an octal digit according to Table I.

Table I
Binary—Octal Conversion

Binary	000	001	010	011	100	101	110	111
Octal	0	1	2	3	4	5	6	7

We convert the resultant octal number to decimal form by adding the octal digits, $d_1, d_2, d_3, \cdots$, each multiplied by the appropriate negative power of 8. Rounded negative powers of 8 are listed in Table II.

The dial of a desk calculator upon which the product appears will be referred to as the "accumulator." The dial upon which the multiplier is recorded will be called the "counter."

We illustrate the mechanics of our process for binary-decimal conversion by converting to decimal form the 43-binary-digit number

$$N = +.110\ 010\ 010\ 000\ 111\ 111\ 011\ 010\ 101\ 000\ 100\ 010\ 000\ 101\ 1.$$

TABLE II

Negative Powers of 8

Unshifted	Shifted
$1/8 = .125$	$1/8 = .125$
$1/8^2 = .01562\ 5$	$10/8^2 = .15625$
$1/8^3 = .00195\ 3125$	$10^2/8^3 = .19531\ 25$
$1/8^4 = .00024\ 41406\ 25$	$10^3/8^4 = .24414\ 0625$
$1/8^5 = .00003\ 05175\ 7812$	$10^4/8^5 = .30517\ 57812$
$1/8^6 = .00000\ 38146\ 9727$	$10^5/8^6 = .38146\ 9727$
$1/8^7 = .00000\ 04768\ 3716$	$10^6/8^7 = .47683\ 716$
$1/8^8 = .00000\ 00596\ 0464$	$10^7/8^8 = .59604\ 64$
$1/8^9 = .00000\ 00074\ 5058$	$10^8/8^9 = .74505\ 8$
$1/8^{10} = .00000\ 00009\ 3132$	$10^9/8^{10} = .93132$
$1/8^{11} = .00000\ 00001\ 1642$	$10^5/8^{11} = .00001\ 1642$
$1/8^{12} = .00000\ 00000\ 1455$	$10^6/8^{12} = .00001\ 455$
$1/8^{13} = .00000\ 00000\ 0182$	$10^7/8^{13} = .00001\ 82$
$1/8^{14} = .00000\ 00000\ 0023$	$10^8/8^{14} = .00002\ 3$
$1/8^{15} = .00000\ 00000\ 0003$	$10^9/8^{15} = .00003$

Convert N to its 15-digit octal equivalent by inspection,

$$N = +\ .62207\ 73250\ 42054.$$

0) Clear the calculator. Enter the 10 most significant digits (.62207 73250) of the octal equivalent in the counter. (On calculators with automatic multiplication this may be done by executing .62207 73250 × .00000 00000; otherwise it must be done digit by digit.) Shift the carriage to the extreme right. Set the counter lever to count subtractions. Set the keyboard lever to hold the number until manually cleared. Assume the keyboard decimal point to be on extreme left. Assume the accumulator decimal point to be immediately above keyboard decimal point. This places the most significant octal digit above the extreme right position of the keyboard.

1) $1/8 = .12500\ 00000$ to keyboard; add until 1st counter digit is zero; shift 1 left; and clear keyboard.

2) $10/8^2 = .15625\ 00000$ to keyboard; add until 2nd counter digit is zero; shift 1 left; and clear keyboard.

. . . . .

10) $10^9/8^{10} = .93132\ 00000$ to keyboard; add until 10th counter digit is zero. (The accumulator contains .78539 81629 0139 = 10th partial decimal sum.)

10′) Clear keyboard. Enter five least significant digits of $8^5 \times$ (octal equivalent)—i.e., .00000 42054 in counter. Shift carriage four positions to the right.

11) $10^5/8^{11} = .00001\ 16420$ to keyboard; add until 6th counter digit is zero; shift 1 left; and clear keyboard.

. . . . .

15) $10^9/8^{15} = .00003\ 00000$ to keyboard, and add until 10th counter digit is zero. (Accumulator contains .78539 81633 9744 = 14-decimal-digit equivalent of given binary number $\cong \pi/4$.)

In the above process, we added to the $(j-1)^{st}$ partial decimal sum a prescribed number of multiples of $1/8^j$. In the inverse process, we subtract from the $(j-1)^{st}$ partial decimal remainder as many positive integral

multiples of $1/8^j$ as possible. Specifically, enter all 14 decimal digits in the left hand end of the accumulator and shift carriage to the extreme right. Now subtract out multiples of $1/8^j$ starting with $j = 1$ and shifting carriage one left after each subtraction. When the 10 most significant octal digits have been obtained in this way, the counter will be filled. Clear counter, shift carriage 4 right. Continue the subtracting and shifting cycle until the 5 least significant octal digits have been obtained.

The process of converting from 43 binary digits to 15 octal digits is exact. In converting from 15 octal digits to 14 decimal digits, an error, $\epsilon_D = \sum_{j=1}^{15} d_j\epsilon_j$, is introduced, where ϵ_D is the error of the resultant decimal equivalent, d_j is the j^{th} octal digit to the right of the octal point, and ϵ_j is the error of the tabular approximation to the quantity $1/8^j$. (All errors are taken in the sense that the true value is equal to the approximate value plus the error.) Comparing the j^{th} entry of Table II with the exact value of $1/8^j$ we find that

$$\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = 0$$

$$\begin{aligned}
\epsilon_6 &= -.44 \times 10^{-14} & \epsilon_5 &= +.50 \times 10^{-14} \\
\epsilon_7 &= -.18 \times 10^{-14} & \epsilon_8 &= +.48 \times 10^{-14} \\
\epsilon_{11} &= -.47 \times 10^{-14} & \epsilon_9 &= +.06 \times 10^{-14} \\
\epsilon_{13} &= -.11 \times 10^{-14} & \epsilon_{10} &= +.26 \times 10^{-14} \\
\epsilon_{14} &= -.27 \times 10^{-14} & \epsilon_{12} &= +.20 \times 10^{-14} \\
\epsilon_{15} &= -.16 \times 10^{-14} & &
\end{aligned}$$

Now $d_j = 0, 1, \cdots, 7$ for $j = 1$ to 14 and $d_{15} = 0$ or 4. Therefore, ϵ_D is greatest when $d_6 = d_7 = d_{11} = d_{13} = d_{14} = d_{15} = 0$ and $d_5 = d_8 = d_9 = d_{10} = d_{12} = 7$. On the other hand, ϵ_D is least when $d_6 = d_7 = d_{11} = d_{13} = d_{14} = 7$, $d_{15} = 4$, and $d_5 = d_8 = d_9 = d_{10} = d_{12} = 0$. Thus $-10.93 \times 10^{-14} \leq \epsilon_D \leq 10.50 \times 10^{-14}$.

In converting from 14 decimal digits to 15 octal digits, a remainder, R, is left in the accumulator after the 15 octal digits have been obtained. But R is in error by an excess of $\sum_{j=1}^{15}{}' d_j\epsilon_j$, where the prime after the summation symbol denotes that now d_{15} as well as d_1 through d_{14} may range over all positive integers between 0 and 7, inclusive. Therefore, the error, ϵ_0, of the octal equivalent is given by

$$\epsilon_0 = R - \sum_{j=1}^{15}{}' d_j\epsilon_j$$

or

$$R_{\min} - \left(\sum_{j=1}^{15}{}' d_j\epsilon_j\right)_{\max} \leq \epsilon_0 \leq R_{\max} - \left(\sum_{j=1}^{15}{}' d_j\epsilon_j\right)_{\min}.$$

But, from Table II, $0 \leq R \leq 3 \times 10^{-14}$. Therefore,

$$0 - (10.50 \times 10^{-14}) \leq \epsilon_0 \leq (3 \times 10^{-14}) - (-11.41 \times 10^{-14})$$

or

$$-3.70 \times 8^{-15} \leq \epsilon_0 \leq 5.07 \times 8^{-15}.$$

Conversion of the 15 octal digits to 43 binary digits is exact except for the final rounding of the least significant octal digit. This rounding introduces an error of at most 1×2^{-44} in absolute value. For the total error, ϵ_B, of the binary equivalent of the original decimal number we have then

$$(-3.70 \times 8^{-15}) + (-1 \times 2^{-44}) \leq \epsilon_B \leq (5.07 \times 8^{-15}) + (1 \times 2^{-44})$$

or

$$-1.43 \times 2^{-43} \leq \epsilon_B \leq 1.77 \times 2^{-43}.$$

The error of the binary equivalent might be reduced further by rounding the remainder left in the accumulator instead of neglecting it. The additional accuracy to be gained, however, is insufficient to justify complicating the procedure.

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[1] Florence Koons & Samuel Lubkin, "Conversion of numbers from decimal to binary form in the EDVAC," *MTAC*, v. 3, p. 427–431.

Bibliography Z-XIII

1. W. Barrois & R. Basile, "Le calcul matriciel et l'aviation," *Atomes*, v. 5, no. 50, 1950, p. 165–169, figs.

The solution of vibration problems in airplane design is discussed by the authors. A brief statement of the nature and importance of the vibration problem is followed by an elementary treatment of the method of reducing it to a form suitable for solution on punched-card equipment. The economy in presentation of the problem by use of the matrix notation is pointed out. The utility of the matrix calculus in solving the problem is indicated.

Some computational results obtained on punched-card equipment are listed. The solution of a system of six equations in six unknowns required one hour and thirty minutes; that of eight such systems required six hours. Eight systems of eight equations in eight unknowns required nine hours. The product of two eighth-order matrices was computed in one hour and fifty minutes. Two such matrix products were computed in two hours and thirty minutes. In regard to the complete vibration problem, the authors state that the use of punched-card equipment increased the computation rate by a factor of four to eight.

E. W. C.

2. Edmund C. Berkeley, *Giant Brains or Machines that Think*, John Wiley & Sons, Inc., New York, and Chapman & Hall, Ltd., London, 1949, 270 + xvi p., 14 × 21.2 cm., bibl., figs., $4.00.

This is an interesting book, which is designed for the general reader. In this essay in scientific journalism the author has been very successful in keeping explanations and language simple.

Chapter 2 deals with "Languages"—systems for handling information, with particular emphasis on numerical information, while Chapter 3 outlines the principles of design of automatic machines by detailing the design of a very simple model using a finite arithmetic of four numbers, with four operations: addition, negation, greater than, and selection.

In Chapters 4 to 9, machines in existence up to the end of 1946 are described, i.e., punched-card machines, the MIT differential analyzer No. 2, the Harvard IBM Automatic Sequence-Controlled Calculator, the ENIAC, the Bell Laboratories General-Purpose Relay Calculator, and the Kalin-Burkhart Logical Truth Calculator. The origin and organization of each machine are described as are the processes which it can carry out. Finally an appraisal of each machine is given.

Chapter 10 describes further developments in progress when the book was written—mercury delay lines, Williams tubes, magnetic tapes, etc., for storage, new operations, and new ideas in programming and reliability. Machines under construction are also described briefly; several have been subsequently put into operation.

Chapter 11 describes possible future developments—some far-reaching—and should stimulate a good deal of thought. Some of these are automatic library, automatic translator, automatic typist and stenographer, automatic recognizer, and so on. Problems possibly suitable for mechanization, such as problems of control, weather, psychological testing and training, business, etc., are suggested and discussed.

In Chapter 12 the problem of social control is discussed, and the difficulties that may be caused by the advent of automatic machines on a large scale are enumerated.

The major criticism of the reviewer concerns the title and subtitle of the book. The phrases, "giant brains" and "machines that think," may stimulate interest in those new to the ideas involved in automatic machinery, but may be dangerous and can only irritate most of those who really study the machines and processes involved. The giantness is a measure of inefficiency, the much better human brain is much more compact. Chapter I in the book, deliberately mentioned last in this review, is a discussion on "Can machines think?" which is answered in the affirmative. This is only because the author defines "thinking" in such a way that the answer *can* be in the affirmative. To be quite fair one must mention that the author lists "the kinds of thinking a mechanical brain can do" (all can be classed as instinctive or "thoughtless" reactions) and the kinds of thinking it cannot do—the important kinds! The objection to these ideas is that the reader will not realize that *all* important kinds of thinking are excluded—only automatic reactions remaining. This could be dangerous.

This criticism is mentioned mainly because the first chapter is involved, and this may prevent many readers from continuing, which could be a pity, for the book is a stimulating account.

J. C. P. Miller

NBSNAML

3. Tung Chang Chen, "Diode coincidence and mixing circuits in digital computers," I. R. E., *Proc.*, v. 38, 1950, p. 511–514.

Two types of gate circuits, the coincidence and mixing, are derived in terms of germanium diodes. The operation of the coincidence gate is analyzed for transient response with a rectangular pulse input taking into account the "back" resistance and capacity of the coupling and clamper diodes. The operation of a typical mixing gate using germanium diodes is analyzed for transient output for a rectangular pulse input under the conditions stated

above. An equivalent circuit is described of a driving source and one of the inputs of a coincidence gate for positive pulses. The article concludes by illustrating and listing some uses of these gates in digital computer circuits.

M. M. ANDREW

NBSMDL

4. J. P. ECKERT, JR., H. LUKOFF, & G. SMOLIAR, "A dynamically regenerated electrostatic memory system," I. R. E., *Proc.*, v. 38, 1950, p. 498–510, bibl.

This comprehensive paper on electrostatic memory systems discusses various techniques of using commercial cathode-ray tubes as electrostatic storage elements. After a brief introductory section on the fundamental theory of the subject, the authors list and describe seven possible methods of storing binary information in cathode-ray tubes. Included among these methods are the dot-line and the dot-circle techniques. After a discussion of these various storage methods, the authors conclude, from the characteristics of the reading voltage, the speed of reading, and the signal stability obtained, that the dot-circle method is the best of those examined. However, other investigators in this field using different methods for utilizing the dot-line method of storage may challenge their conclusion here.

Tests of the commercial phosphors as functions of high output signal at high accelerating voltages, ease of erasure, and cost were made. It was concluded that P1 phosphor is the best of the commercial phosphors for this specialized use.

Using the dot-circle method of storage and P1 phosphor surfaces, the authors carried through a series of tests in which the following factors were among those investigated:

1. Gun structure
2. Tube diameter (3″, 5″, 7″)
3. Accelerating voltage
4. Grid voltage
5. Input circuit design.

Their research on the effect of tube diameter versus the number of storage spaces brings out the interesting point that the increase in number of spots stored on a 7″ tube over the number stored on a 3″ tube is only about 73%, while the increase in area is about a factor of 5.

The remainder of the article is devoted to a discussion of regeneration and deflection circuits and the use of these tubes in memory units of digital computers operated in parallel or serial modes.

M. M. ANDREW

NBSMDL

5. W. B. FLOYD, "Electronic machines for business use," *Electronics*, v. 23, May 1950, p. 66–69.

A businessman expresses his opinion on the applicability of electronic computers to the clerical work of business. In most cases, it is stated, sorting, lookups, posting, and typing far outweigh the arithmetical work in a clerical

procedure. In business applications, a computer would in general perform relatively simple mathematical operations on a vast amount of data.

Special design features which would be necessary in an ideal clerical computer are discussed. The elimination of a manual keyboard, or at least the reduction of manual key depressions to a minimum, is recommended. A step in this direction has already been made by the retail garment trade, in the development of equipment for perforating as well as printing code numbers on marketing tickets. Extremely fast output printers would be necessary; over 1,000 printed lines a minute might be a usable speed. Storage capacity undreamed of in connection with mathematical computers would be needed. Requirements for the storage in readily accessible form of hundreds of thousands of items would not be unusual. An alleviating factor regarding storage is that access times to computer memory would not need to be comparable to those deemed essential for mathematical computations.

The attractive features of electronic digital computers, from the standpoint of business applications, are given as the following. It is unnecessary to perform a series of separate mechanical operations to produce a single result. Interposition of manual operations during a computation is reduced to a minimum. The electronic machines, because of their selective sequencing feature, can recognize and handle all irregularities the possibility of which can be foreseen by the human operator. Whatever rules can be given to a clerk can be included in the programmed machine instructions.

The author states that all of the computer circuits required for a useful clerical computer are at hand and have been well proven. What remains to be done, in his opinion, is industrial engineering—the engineering of electronic computing equipment for particular business applications.

E. W. C.

6. Office of Naval Research, *Digital Computer Newsletter*, v. 2, no. 2, May 1950, 4 p.

This is the fourth release of the Mathematical Sciences Division of the ONR. Previous releases are dated April 1949, Sept. 1949, Jan. 1950. The present status of the following digital computer projects is treated briefly in this number.

1. Naval Proving Ground Calculators
2. Raytheon- Computers
3. UNIVAC
4. Aberdeen Proving Ground Computers
5. The California Digital Computer
6. Institute for Advanced Study Computer
7. Project Whirlwind
8. MADDIDA Computer
9. Institute for Numerical Analysis Computer
10. NBS Computer
11. Computers, Manchester University, England
12. Telecommunications Research Establishment Computer
13. BARK Computer, Sweden.

7. LOUIS N. RIDENOUR, "High-speed digital computers," *Jn. Applied Physics*, v. 21, 1950, p. 263–270.

This is a survey article which discusses current machine design, capabilities, and future trends in the computer field. Of particular interest is the discussion of the application of the Monte Carlo method by these machines.

8. CLAUDE E. SHANNON, "A chess-playing machine," *Scientific American*, v. 182, Feb. 1950, p. 48–51.

A machine is proposed which can play a reasonably good game of chess at speeds comparable to human speeds. The author states that it is impossible to achieve perfection in a machine game as even the high-speed electronic computers cannot calculate all the possible variations to the end of the game. This seemingly trivial investigation is undertaken to develop techniques which can be used for more practical applications.

Some of the empirical methods used by chess experts are programmed for the machine, which examines only the important possible variations in the game in sufficient detail to make clear the consequences of a particular move. The machine advantages are listed as: much greater speed, freedom from error (except for those due to programming deficiencies), freedom from laziness, and freedom from "nerves." Human advantages are flexibility, imagination, and learning capacity.

The ability of these machines to think depends upon the definition of thought. The computer follows the strictly behavioristic patterns which, according to some psychologists, characterize thought. However, the author points out the more important fact that the machine is unable to learn by its mistakes and it is wholly dependent upon the programmer who outlines its course of action.

EDITH NORRIS

NBSMDL

NEWS

American Institute of Electrical Engineers.—The Summer and Pacific General Meeting was held June 12 through 16, 1950, at Pasadena, California. One morning section on June 13 was devoted to discussions of large-scale computers and an afternoon section on the same day had as its topic "Applications of computers to aircraft engineering problems."

The program for the morning section was as follows:

"Design features of the National Bureau of Standards Western Automatic Computer," by E. LACEY, D. RUTLAND, H. LARSON, & H. D. HUSKEY, NBSINA.

"Applications of the National Bureau of Standards Western Automatic Computer," by H. D. HUSKEY, NBSINA.

"University of California Digital Computer," by P. A. MORTON, Univ. of Calif.

"A high-speed multiplier for analog computers," by B. N. LOCANTHI, Calif. Inst. of Tech.

"MADDIDA (Magnetic Drum Digital Differential Analyzer), general theory," by F. G. STEELE, Northrop Aircraft.

"MADDIDA, design features," by D. E. ECKDAHL, Northrop Aircraft.

In the afternoon the following talks were presented:

"Automatic data handling techniques, including recording and reduction," by W. D. BELL, Telecomputing Corp.

"Electric analog computing techniques for complex vibration and aeroelastic problems," by G. D. MCCANN & R. H. MACNEAL, Calif. Inst. of Tech.

"Use of analogy computing techniques for aeroelastic problems," by P. A. Dennis & D. G. Dill, Douglas Aircraft.

"Complex missile control system, design, and analysis with the electric analog computers," by J. P. Brown, Lear, Inc., & C. H. Wilts, Calif. Inst. of Tech.

"Solution of problems in electrical engineering by means of analog computers," by L. L. Grandi & D. Lebell, Univ. of Calif.

National Bureau of Standards.—In June 1950, the NBS Eastern Automatic Computer, called SEAC, was formally dedicated as an operating computer. (See *MTAC*, v. 4, p. 164–168.) Prior to its dedication, SEAC had solved: (1) miscellaneous mathematical exercises such as determination of prime numbers, computation of sine-cosine tables, solution of diophantine equations; (2) a skew-ray problem for the NBS Optics Division; (3) a problem concerning the flow of heat in a chemically reactive material; and (4) an initial problem for Project SCOOP (Scientific Computation of Optimum Programs) for the Office of the Air Comptroller, Department of the Air Force.

SEAC will be used to solve scientific problems for the NBS and production-scheduling problems for the Air Force. It will also serve as an instrument for evaluating the effectiveness and reliability of computer components, and it will increase present knowledge of the maintenance and servicing problems related to computers.

SEAC was completed 14 months after construction of the machine was undertaken. Most of the work on the computer (the design, engineering, fabrication, and assembly) was performed by the NBS staff in its Washington laboratories. The only phase of the work not accomplished by the Bureau staff was the fabrication of the acoustic memory unit of the machine, which was carried out by the Technitrol Engineering Company, Philadelphia, Pennsylvania.

Simon, a small-scale computing machine.—On Thursday, May 18, 1950, this small computer was unveiled at Columbia University. This tiny machine was conceived by Edmund C. Berkeley, actuary and consultant member of Connell, Price and Co., and is described in his book, *Giant Brains, or Machines that Think*, [*MTAC*, v. 4, p. 234.] It is intended to be used primarily for teaching purposes to stimulate thinking and understanding and to produce training and skill. The machine represents the combined efforts of technician William A. Porter and of electrical engineers Robert A. Jensen and Andrew Vall. This low cost machine is 24 inches long, 15 inches wide, 6 inches thick, and weighs 39 pounds. It will perform the operations of addition, subtraction, greater than, and selection employing an arithmetic of four numbers.

OTHER AIDS TO COMPUTATION

Bibliography Z-XIII

9. D. P. Adams, *An Index of Nomograms*. The Technology Press of MIT and John Wiley & Sons, New York, 1950. v + 174 p. 18.2 × 24.1 cm. $4.00.

This is another example of the recent trend of compiling references in a specialized field thus hoping to cope with the vast extent of present day scientific activity. The present index contains over 1,700 references to nomograms which have appeared since 1923 in 97 selected journals. The references are listed under 21 main headings and are extensively cross referenced by key words. Since the equations are not given, however, there is not much chance of adapting a nomogram from one field to another unless the reader is well acquainted with both.

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10. H. Billing, "Numerische Rechenmaschine mit Magnetophonspeicher," *Zeit. angew. Math. Mech.*, v. 29, 1949, p. 34–36.

A magnetic drum memory for 129 numbers, of 20 binary digits each, is described.

11. S. L. Brown & C. M. McKinney, "Use of mechanical harmonic synthesizer in electrical network analysis," *Jn. Appl. Phys.*, v. 20, 1949, p. 316–318.

This paper gives two examples of the use of a harmonic synthesizer to compute the real and imaginary components of a complex AC impedance as a function of frequency. Each of these components can be expressed as the quotient of two polynomials in w. By using a change of variable of the form, $w = A + B \cos \theta$, the numerators and denominators of these functions are put in the form of a Fourier series, which is then summed by the use of the synthesizer. The quotient is then the desired result. The constants A and B determine the range of values of w for which the values of the components are obtained. The synthesizer is used only to compute the value of two polynomials. The computations required by the change of variables and the necessary division must be done by other means. If a harmonic synthesizer is available, its use will result in a considerable saving of time. However, being an analogue device, the resulting accuracy is limited because of the usual scale factors.

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12. S. L. Brown & J. M. Sharp, "Use of a mechanical harmonic synthesizer in electric wave filter analysis," *Jn. Appl. Phys.*, v. 20, 1949, p. 578–582.

This paper gives an example of the use of a harmonic synthesizer to compute the attenuation, phase-shift, and impedance characteristics of a filter having two pass bands. Each of these characteristics depends on frequency functions which can be expressed as the quotient of two polynomials in w. The procedure is similar to that described in the previous review.

J. B. Russell

13. Jules Lehman, "Harmonic analyzer and synthesizer," *Electronics*, v. 22, Nov., 1949, p. 106–110.

The instrument described in this paper is based on the use of a set of suitably geared synchrotransformers. A linear network is used to obtain the sine and cosine from the three phase output. The main application contemplated is the transformation of a frequency response curve to a square wave response curve and vice versa.

F. J. M.

14. A. S. Levens, *Nomography*. John Wiley & Sons, Inc., New York, Chapman & Hall, Ltd. London, 1948. vi + 176 p. 15.2 × 22.9. Price $3.00.

The book contains an elementary treatment of nomography with numerous illustrative examples. The first twelve chapters are devoted mainly to

special types of nomograms while the thirteenth gives a brief description of the general determinantal theory of nomograms.

R. W. HAMMING

15. W. P. LINTON, "Nomographic analysis of rectangular sections of reinforced concrete," Amer. Soc. Civil Engineers, *Proc.*, v. 75, 1949, p. 129–142.

This paper contains four nomograms for use in the design of rectangular sections of reinforced concrete. A careful discussion of the use of these nomograms is given. Appendix 1 gives the derivations of the formulae and appendix 2 gives the design methods used in constructing them.

R. W. HAMMING

16. W. A. MCCOOL, *DC Analog Solution of Simultaneous Linear Algebraic Equations: Circuit Stability Considerations*, NRL report 3533, Naval Research Laboratory, Washington D. C., 1949, iv + 10 p.

This report describes a process of solving simultaneous linear equations on analogue equipment in which the feedback is arranged on an equation by equation basis if necessary in order to obtain a stable result. A purely linear feedback, which is stable for all systems of equations must involve the coefficients of the given system[1] and previous proposals for stable setups in general involved using the full matrix twice.[2] However, if we have a given fixed feedback setup, i.e., fixed in the sense that the coefficients are not involved, the criterion for stability can be regarded as dividing the space of matrices in to 2^n parts, in only one of which, we will have stable systems of equations. Now it is possible to vary the signs of the feedback for the unknown in 2^n ways and by certain other adjustments to obtain stability without entering the values of the coefficients into the feedback directly. Consequently the amount of equipment used in this setup is only one-half that necessary for the duplicate matrix procedure.

With minor simplifications the procedure for obtaining stability is the following. By obvious rearrangements and changes of sign one can suppose that the diagonal elements a_{jj} are positive and $a_{jj} \geqq a_{ll}$ if $l \geqq j$. First let us consider a feedback setup in which each x_j is given by

$$x_j = -\mu(a_{j1}x_1 + \cdots + a_{jn}x_n - y_j),$$

where μ is of course the amplifier gain. The μ is a function of p but, normally, is considered to be large and positive. The arrangement of the system given above seems to give the best chance for stability and a system involving one equation and one unknown would be stable. On the other hand, it is not necessarily stable and the author proposes the following modification in this case.

Suppose that the system obtained from the first $j - 1$ equations by suppressing the last $n - j + 1$ unknowns is stable. This means that the determinantal equation for this matrix obtained by replacing λ in the characteristic equation by $-1/\mu(p)$ has roots with only negative real parts. Suppose then that the solution obtained by introducing the j'th equation and the j'th unknown is unstable when the above feedback is used. In this

case the amplifier previously used is replaced by an integrating amplifier so that the feedback is given by

$$(a_{j1}x_1 + \cdots + a_{jj}x_j) = \pm RCpx_j.$$

Here RC is large. When the stability equation is written out in polynomial form the large factor RCp insures that in general all but one of the roots are close to the corresponding roots of the stable $j - 1$st order system. The remaining root is small and can be made to have a negative real part by properly choosing the sign of the integrating amplifier output, when RCp dominates the situation adequately.

If RC is fixed, there will be systems which would not respond to the above, although for large RC, this possibility may be negligible. On the other hand, if RC is adjustable any system can be stabilized. Adjusting RC is probably also desirable for accuracy reasons, although this is not given by the author. The methods by which stability are obtained, in general, decrease the accuracy of the result. For instance, the use of a double network squares the factor φ by which one multiplies an error in the equations to obtain the corresponding error in the unknowns. The reviewer is under the impression that after the equations have been rearranged, the smaller the concessions made for stability, the more accurate will be the result. For the larger RC is taken the smaller will be the least characteristic root of the stability equation and φ is in general the reciprocal of this root.

The McCool stabilizing technique appears to be a highly significant generalization of the Goldberg & Brown feedback method.

F. J. M.

[1] F. J. Murray, "Linear equation solvers," *Quart. Appl. Math.*, v. 7, 1949, p. 263–274.
[2] E. A. Goldberg & G. M. Brown, "An electronic simultaneous equation solver," *Jn. Appl. Phys.*, v. 19, 1948, p. 339–345, or F. J. Murray, *The Theory of Mathematical Machines*, New York, 1947, p. 92 (1st ed.), p. III, 20–21 (2nd ed.).

17. W. Meyer zur Capellen, *Mathematische Instrumente* (*Mathematik und ihre Anwendungen in Physik und Technik*, ed. by E. Kamke & A. Kratzer, s. B, v. 1). Third enlarged ed., Leipzig, Akademische Verlagsgesellshaft, 1949, x, 339 p.

A review of an American reprint of the second edition of this book appeared in *MTAC* [v. 3, p. 137–138]. We are told that most of the present edition was in type in early 1945. Twenty-six new pages have been added. A second supplement to the literature list adds entries 311–346, with only two dated later than 1945 (1947, 1948). Then follow a number of supplements to previous paragraphs on various small machines, and, p. 311–315, new paragraph about Zuse's machine (*MTAC*, v. 2, p. 355–359, 367–368), Mark I, ENIAC, and ACE. Harmonic analyzers, the differential equation machine at the Institute for Practical Mathematics at Darmstadt (1938–1947), and other machines are discussed in the final text pages 315–331. The new name and subject index includes references to the new material.

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18. R. SAUER, *Über den Entwurf von Schaltungen der Universal-Integriermaschine*, Institut für praktische Mathematik, Ummendorf Würt, [Translation: *Design of Circuit Controls for the Universal Integrating Machine*, Air Documents Division, T-2, AMC, Wright Field, Microfilm No. RC-905, F 8766.] (Review from copy in Brown Univ. Library.)

This report contains a detailed analysis of the causes of errors in differential analyzers and a rough method for estimating the error. In addition, details and examples are given for the process of setting up "flow diagrams," scale determinations and the use of amplifiers in a differential analyzer. These discussions supplement the work of PÖSCH and SAUER available in the literature.[1]

F. J. M.

[1] H. PÖSCH & R. SAUER, "Integriermaschine für gewöhnliche Differentialgleichungen," Verein Deutsch. Ingen., *Zeit.*, v. 87, 1943, p. 221–224.

19. K. SPANGENBERG, G. WALTERS, & F. SCHOTT, "Electrical network analyzers for the solution of electromagnetic field problems," I.R.E., *Proc.*, v. 37, 1949, p. 724–729, 866–873.

Two networks are described for obtaining solutions of the wave equations, based on the ideas developed by KRON. One network was set up for two dimensional cylindrical coordinates and the values for the network elements for the TM_0, TEM, and TE_0 modes are given. The second network corresponds to two dimensional rectangular coordinates and the values for the TE and TM modes are given. Detailed design considerations are given including the problem of choice of frequency, permissible range of Q for the coils and the physical set up. A frequency range 20 to 300 kc. was used. In the first network a fine section for greater detail was used and the matching problem is discussed. The total cost is also given.

Part II discusses the tuning of the networks and the uses of these networks which are applied to determine cavity resonance frequencies, impedances and Q. Accuracy of the results is checked by taking the difference equations corresponding to the network and substituting in and then by an iterative process obtain the solution of these equations.[1]

F. J. M.

[1] Cf. G. H. SHORTLEY & R. WELLER, "Numerical solution of Laplace's equation, *Jn. Appl. Phys.*, v. 9, 1948, p. 334–348.

20. A. WALTHER, "Lösung gewöhnlicher Differentialgleichungen mit der Integrierslage IPM—Ott," *Zeit. angew. Math. Mech.*, v. 29, 1949, p. 37.

The differential analyzer described uses a "steering wheel" integrator as proposed by U. KNORR. It is located at Institut für Praktische Mathematik of the Technische Hochschule, Darmstadt.

F. J. M.

21. A. WALZ, "Ein waagähnliches Gerät für harmonische Analyse und Synthesis," *Zeit. angew. Math. Mech.*, v. 29, 1949, p. 42–44.

Consider a bar symmetrically mounted on a horizontal axle. Let a mass M be located at a distance l from the axle along this bar. If the axle is turned

on amount ϕ from the equilibrium position, a turning moment of amount $Ml \sin \phi$ appears. Such moments are readily added and can be used to realize expressions in the form $\sum_a Ml_a \sin a\phi$, and $\sum_a Ml_a \cos a\phi$, upon which harmonic analysis and synthesis can be based.

F. J. M.

22. H. WITTKE, "Mathematischen Maschinen und Instrumente vom Abacus zum Eniac," *Zeit. angew. Math. Mech.*, v. 29, 1949, p. 34–36.

This summary contains a list of dates in the development of computing machines, a list of European desk machines and a list of present large scale computers or computing projects.

23. K. ZUSE, "Die Mathematischer Voraussetzungen für die Entwicklungen logistiche kombinativer Rechenmaschinen," *Zeit. angew. Math. Mech.*, v. 29, 1949, p. 36–37.

A brief description of the fundamental theory needed for the development of a "logistic" computer and of the author's proposal for this theory based on the propositional calculus.

NOTES

120. TESTS OF RANDOM DIGITS.—KENDALL & BABINGTON-SMITH[1] have described four tests of local randomness to be applied to any set of locally random digits. These four are (a) the frequency test, (b) the serial test, (c) the gap test, and (d) the poker test. It is the purpose of this note to show how these tests may be applied to any set of digits, punched on IBM cards, mechanically and without regard to the order of the digits on the cards, using standard IBM equipment.

Kendall and Smith applied these four tests to their table[2] of 100,000 random digits, by hand, taking the digits in the order in which they are printed in the table.

The frequency test consists in counting the frequency of occurrence of each of the ten digits, with expected values of 10% for each digit. This test may most easily be made on the sorter, provided it is equipped with the card-counting device, counting the cards that fall into each pocket when sorting on any one column. Alternatively, a tabulator equipped with digit selectors can effectively sort any two columns simultaneously and print the tabulation at the end of a run of cards.

The serial test is essentially a frequency test of two-digit numbers, with each two-digit combination from 00 to 99 expected to occur 1% of the time. The cards can be sorted on any two columns first and then, with the tabulator controlling on those two columns, a card count will record the frequencies.

The gap test, as described by Kendall, consists in counting the gap between successive zeros in the table. As done by hand, the test is applied to the digits horizontally, row by row. Using punched cards, it is easier to apply the test vertically through the table, working down through the columns. If the cards are serially numbered, they can be sorted on any column and the serial numbers of the cards falling in the zero pocket reproduced onto another deck. First differences are then taken of these serial

numbers, which are the required gaps. The cards are then restored to order, and the zeros sorted out on another column, and the process repeated, column by column. It is also possible to count the gaps between zeros directly on the tabulator by wiring from the upper brushes to a comparing relay and having the resulting unequal impulses pick up a selector which directs a counter, continuously adding the card count impulse, when to take a total. The comparing relays of the tabulator respond only to digits other than zero. With this latter method, many columns of the cards may be tested for gaps between zeros simultaneously.

The poker test consists in grouping any four digits into one of five types like poker hands: four of a kind, three of a kind, two pairs, one pair, and none alike. It is this test which is most easily applied with a tabulator. Any four columns (call them A, B, C, and D) are wired from both upper and lower brushes to six comparing relays, as follows: from upper brushes B B B C C A to six relays in that order, and from lower brushes, C A D A D D. Each of the four columns is thus compared with each of the others. The six unequal impulse hubs pick up six 10-position class selectors. There are only 15 combinations which can arise; the card count impulse is sent directly to five counters and "plug to C" is filtered through the selectors to indicate which of the five counters shall add.[3]

A tabulation on 1,000 cards (bearing 40,000 digits from Kendall and Smith's table), using ten random combinations of four columns, is as follows:

4 of a kind	14
3 of a kind	370
two pairs	279
one pair	4,258
none alike	5,079

for which a chi-square test shows $p = .50$, approximately, taking 10, 360, 270, 4,320, and 5,040 as the expected values, respectively.

It is probably possible, but not convenient, to extend such a test to 5-digit poker hands (having, in addition to the five types mentioned, "5 of a kind" and "full house," of the form a a a b b). However, it is relatively easy to analyze the 5-digit numbers found by reading down a column of the table; that is, taking five successive cards on the same card column. The tabulator can be set to take a total every five cards and print, using the digit selectors, a distinctive combination of ones for each of the seven possible poker "hands." These combinations can then be keypunched and tabulated separately.

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[1] M. G. Kendall & B. Babington-Smith, "Randomness and random sampling numbers," R. Stat. Soc., *Jn.*, v. 101, 1938, p. 147–166; "Second paper on random sampling numbers," R. Stat. Soc., *Jn.*, *Supplement*, v. 6, 1939, p. 51–61.

[2] M. G. Kendall & B. B. Smith, "Tables of random sampling numbers," *Tracts for Computers*, No. 24, Cambridge University Press, 1939.

[3] The wiring diagram for the 4-digit poker test is available on request from the Computing Service, North Hall, University of Wisconsin, Madison. A tabulator, type 405 or 416, equipped with at least the selector capacity described, is needed. The wiring diagram shows the wiring for a machine equipped with six 10-position class selectors.

QUERIES and QUERIES—REPLIES

Editorial Note.—Although the Editors' file of **Q** and **QR** is empty at this time, the absence of these sections from this number of *MTAC* does not indicate that they are being abandoned. The Editors will be glad to continue this service to readers of *MTAC* as long as there are questions to be asked or answered.

CORRIGENDA

V. 3, p. 172, l. 28, *for* 17 *read* 32, l. 29, there are 15 partial quotients missing in the continued fraction *for* $1/\pi$; following 292 *read*

$$1,1,1,2,1,3,1,14,2,1,1,2,2,2,2,1,84,\cdots.$$

V. 4, p. 16, l. −11, *for* The mean value of arithmetic functions *read* The mean values of arithmetical functions.

V. 4, p. 18, l. 13, *for* 424 *read* 425.

V. 4, p. 19, l. 18, *for* Radan read Radau. l. 19, *for* 1890 *read* 1880. l. 20, *for* 500–503 *read* 520–523.

V. 4, p. 23, l. 4, *insert* $\alpha = .75$.

V. 4, p. 24, l. 1, *for* The theory *read* The mathematical theory. l. −11, −12 *for* $\ln x$ *read* $\frac{1}{2}\ln x$.

V. 4, p. 26, l. −12, $\tau = 0(.2)3.8$, *for* $Y = 2.2(.2)3.8$ *read* $Y = .2(.2)2$.

V. 4, p. 95, l. 16, *for* $J_m(z)$ *read* $iJ_m(z)$. l. 17, *for* e_m *read* πe_m. l. −18, *for* $m = 0$ *read* $m = 1$, *for* $m = 1$ *read* $m = 2$. l. −13 *for* E. A. *read* A. E.

V. 4, p. 102, l. −14, *for* argument *s read* arguments.

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